Ma 116c Homework #1

Due Tuesday, April 23rd at 1:00pm

1) Supply the proofs, left out in lecture, of the following facts about the class $\mathbf{WF} = \bigcup_\alpha R(\alpha)$ of well-founded sets.

   i. For all ordinals α, $R(\alpha)$ is transitive.

   ii. For all ordinals $\beta \leq \alpha$, $R(\beta) \subseteq R(\alpha)$.

   Fix $y \in \mathbf{WF}$.

   iii. For all $x \in y$, $x \in \mathbf{WF}$ and $\mathrm{rank}(x) < \mathrm{rank}(y)$.

   iv. $\mathrm{rank}(y) = \sup\{\mathrm{rank}(x) + 1 : x \in y\}$.

   Verify by induction that, for every ordinal α we have

   v. $\alpha \in \mathbf{WF}$ and $\mathrm{rank}(\alpha) = \alpha$ and $R(\alpha) \cap \alpha = \alpha$.

2) Find a set A such that for every $n \in \omega$, $A \cup \bigcup A \cup \bigcup^2 A \cup \ldots \cup \bigcup^n A$ is not transitive.

3) Define $x\mathbf{R}y$ iff $x \in \mathrm{tr\ cl}(y)$. Show that $\mathbf{R}$ is well-founded and set-like on $\mathbf{WF}$ (Hint: $x\mathbf{R}y$ implies $\mathrm{rank}(x) < \mathrm{rank}(y)$). Let $\mathbf{G}$ be the Mostowski collapsing function for $\mathbf{R}$ on $\mathbf{WF}$. Show that $\mathbf{G}(x) = \mathrm{rank}(x)$ for every x.

4) Define $x\mathbf{R}y$ iff $\langle x, 1\rangle \in y$. Show that $\mathbf{R}$ is well-founded and set-like on $\mathbf{WF}$. Let $\mathbf{G}$ be the Mostowski collapsing function for $\mathbf{R}$ on $\mathbf{WF}$. Define $\check{y}$ recursively by

   $$\check{y} = \{\langle \check{x}, 1\rangle : x \in y\}$$

   and show inductively that $\mathbf{G}(y) = y$. Hence $\mathrm{ran}(\mathbf{G}) = \mathbf{WF}$.