

Ma 116c Homework #4

For seniors and grad students: Complete problems #1 and #2. The set is due Thursday, May 30th, at 1pm.

For frosh, sophomores, juniors: Complete all problems. The set is due Thursday, June 6th, at 1pm.

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Definition. Suppose $\mathbb{P}$ and $\mathbb{Q}$ are posets. An *isomorphism* of $\mathbb{P}$ and $\mathbb{Q}$ is a bijection $i : \mathbb{P} \rightarrow \mathbb{Q}$ such that $i(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$ and $\forall p_0, p_1 \in \mathbb{P}$ we have $p_0 \leq p_1$ iff $i(p_0) \leq i(p_1)$. We say $\mathbb{P}$ and $\mathbb{Q}$ are *isomorphic* if there exists an isomorphism $i : \mathbb{P} \rightarrow \mathbb{Q}$.

- 1) Suppose κ is an infinite cardinal and $\lambda \leq \kappa$. Show that $\text{Fn}(\kappa \times \lambda, 2)$ and $\text{Fn}(\kappa, 2)$ are isomorphic.

Hint: Fix a bijection $f : \kappa \times \lambda \rightarrow \kappa$ and use f to find an isomorphism.

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For the next problem, you may take the following facts for granted.

Fact #1: Suppose $i : \mathbb{P} \rightarrow \mathbb{Q}$ is an isomorphism. For $D \subseteq \mathbb{P}$, we have that D is dense in $\mathbb{P}$ iff $i[D]$ is dense in $\mathbb{Q}$, where $i[D]$ denotes the pointwise image of D .

Fact #2: Suppose M is a ctm of ZFC and $\mathbb{P}, \mathbb{Q} \in M$. Suppose further ($\mathbb{P}$ is isomorphic to $\mathbb{Q}$) ^{M} (by absoluteness of the notion of “being an isomorphism,” this means precisely that there is an isomorphism $i : \mathbb{P} \rightarrow \mathbb{Q}$ such that $i \in M$). Suppose G is a $\mathbb{P}$ -generic filter over M . Then $i[G]$ is a $\mathbb{Q}$ -generic filter over M .

Fact #3: Under the hypotheses of the previous fact, we have $M[G] = M[i[G]]$. That is, if $\mathbb{P}$ and $\mathbb{Q}$ are isomorphic posets in M and $i \in M$ is an isomorphism, then the extension of M by a generic filter $G \subseteq \mathbb{P}$ is the same as the extension of M by the corresponding generic filter $i[G] \subseteq \mathbb{Q}$.

- 2) Suppose GCH holds in M , and let $\mathbb{P} = \text{Fn}(\omega_5, 2)^M$. Show that in $M[G]$ we have $2^\kappa = \omega_5$ for all infinite cardinals $\kappa \leq \omega_4$.

Hint: By #1, $\mathbb{P}$ is isomorphic (in M) to $\text{Fn}(\omega_5 \times \kappa, 2)^M$ for all such κ . Now use the fact above and the relevant theorems from class (plus the assumed cardinal arithmetic in M) to justify that 2^κ is both $\geq \omega_5$ and $\leq \omega_5$ for all such κ .

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We now outline a proof that $\diamond$ can be forced over any ground model M . Recall that a $\diamond$ -sequence is a sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ such that

- i. For every $\alpha < \omega_1$, $S_\alpha \subseteq \alpha$,
- ii. For every subset $X \subseteq \omega_1$, $\{\alpha \in \omega_1 : S_\alpha = X \cap \alpha\}$ is stationary in ω_1 .

That is, for every $X \subseteq \omega_1$ the $\diamond$ -sequence correctly determines $X \cap \alpha$ stationarily often. $\diamond$ asserts that a $\diamond$ -sequence exists.

Let $\mathbb{P}_\diamond$ be the poset consisting of countable sequences p that approximate a $\diamond$ -sequence. That is, let $\mathbb{P}_\diamond = \{p : p \text{ is a sequence of sets } \langle S_\gamma : \gamma < \alpha \rangle \text{ of some length } \alpha < \omega_1 \text{ such that } \forall \gamma < \alpha \text{ we have } S_\gamma \subseteq \gamma\}$.

For $p \in \mathbb{P}_\diamond$ we write $l(p)$ for the length of p . We order $p \leq q$ if p extends q , that is if $l(p) \geq l(q)$ and $p \upharpoonright l(q) = q$.

Notational note: if $p = \langle S_\gamma : \gamma < \alpha \rangle$ then $l(p) = \alpha$, and if $p = \langle S_\gamma : \gamma \leq \alpha \rangle$ then $l(p) = \alpha + 1$. In particular, $l(p)$ is the least ordinal α for which p does not specify a subset S_α of ω_1 .

Observe that $\mathbb{P}_\diamond$ is ω_1 -closed, since if $p_0 \geq p_1 \geq p_2 \geq \dots$ is a countable length descending sequence of elements of $\mathbb{P}_\diamond$ then $p = \bigcup_i p_i$ is also an element of $\mathbb{P}_\diamond$ and lies below every p_i .

- 3) Suppose M is a ctm of ZFC, and let $\mathbb{P} = \mathbb{P}_\diamond^M$. It is not hard to check (by a density argument) that in any $\mathbb{P}$ -generic extension $M[G]$ we have $\bigcup G$ is a sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ of length ω_1 . We wish to show that $\bigcup G$ is in fact a $\diamond$ -sequence in $M[G]$.

It is sufficient to check the following. Suppose $\tau, \sigma \in M^\mathbb{P}$ are names such that, in M , for some $p \in \mathbb{P}$ we have $p \Vdash \tau$ is a club in $\check{\omega}_1$ and $\sigma \subseteq \check{\omega}_1$. (So in any extension $M[G]$ by $G \ni p$ we have $C = \tau_G$ is a club in ω_1 and $X = \sigma_G$ is a subset of ω_1 .)

Show that, in M , there is $q \leq p$ such that $q = \langle S_\gamma : \gamma \leq \alpha \rangle$ and $q \Vdash \check{\alpha} \in \tau$ and $\sigma \cap \check{\alpha} = \check{S}_\alpha$.

Hint: We find such a q by induction in M . Let $q_0 = p$. In V , find $G_0 \ni q_0$ that is $\mathbb{P}$ -generic over M . In $M[G_0]$, $C_0 = \tau_{G_0}$ is a club and $X_0 = \sigma_{G_0}$ is a subset of ω_1 . Find $\alpha_0 \geq l(q_0)$ with $\alpha_0 \in C_0$ and let $S_0 = X_0 \cap \alpha_0$.

Note that since $(\mathbb{P} \text{ is } \omega_1\text{-closed})^M$, $\mathbb{P}$ does not add countable subsets of ω_1 , so $S_0 \in M^1$.

Back in M , by the forcing theorem we can find $q_1 \leq q_0$ that forces $\check{\alpha}_0 \in \tau$ and $\sigma \cap \check{\alpha}_0 = \check{S}_0$. If $\alpha_0 \geq l(q_1)$, then extend q_1 to $q = \langle S_\gamma : \gamma \leq \alpha_0 \rangle$ so that $S_{\alpha_0} = S_0$, otherwise arbitrarily. Then we're done. (Why?)

If $\alpha_0 < l(q_1)$ we repeat this process. Let $G_1 \ni q_1$ be generic. In $M[G_1]$, we again have $C_1 = \tau_{G_1}$ is a club and $X_1 = \sigma_{G_1}$ is a subset of ω_1 . Find $\alpha_1 \geq l(q_1)$ in C_1 and now let $S_1 = X_1 \cap \alpha_1$. And so on.

Either at some finite stage we find (in M) some $q_{n+1} \leq q_0$ forcing $\check{\alpha}_n \in \tau$ and $\sigma \cap \check{\alpha}_n = \check{S}_n$ with $\alpha_n \geq l(q_{n+1})$, in which case we can find $q \leq q_{n+1}$ as desired, or the induction continues for infinitely many stages. In the second case, let $q' = \bigcup_n q_n$. Then $l(q') = \alpha$ where $\alpha = \sup \alpha_n$. Show we can find $q \leq q'$ of length $\alpha + 1$ which is as desired.

Commentary: It follows from (3) that $\diamond$ holds in any $\mathbb{P}$ -generic extension $M[G]$. Here's why. In $M[G]$, let $\bigcup G = \langle S_\alpha : \alpha < \omega_1 \rangle$ be the generic sequence. We claim this is a $\diamond$ -sequence (in $M[G]$). To see this, suppose $X \subseteq \omega_1$ (in $M[G]$). We want to show $\{\alpha : X \cap \alpha = S_\alpha\}$ is stationary (in $M[G]$).

Suppose $C \subseteq \omega_1$ is club (in $M[G]$). Find names τ, σ such that $\tau_G = C$ and $\sigma_G = X$, and find $p \in G$ such that, in M , $p \Vdash \tau$ is club and $\sigma \subseteq \check{\omega}_1$. Then (3) (or really, the argument in (3) carried out below any $p' \leq p$) shows that $D = \{q \leq p : \text{for some } \alpha < \omega_1, q = \langle S_\gamma : \gamma \leq \alpha \rangle \text{ and } q \Vdash \check{\alpha} \in \tau \text{ and } \sigma \cap \check{\alpha} = \check{S}_\alpha\}$ is dense below p . Hence there is $q \in D \cap G$. Then in $M[G]$, $\alpha \in C$ and $X \cap \alpha = S_\alpha$, i.e. $\{\alpha : X \cap \alpha = S_\alpha\}$ intersects C . Since X and C were arbitrary, $\{\alpha : X \cap \alpha = S_\alpha\}$ is stationary, as desired.

We showed in 116b that $\diamond$ implies both the CH and the existence of a Suslin tree. Thus in $M[G]$ there is a Suslin tree, whether or not there was such a tree in M .

¹More explicitly: any countable subset $S \subseteq \omega_1$ in $M[G]$ is a subset of some $\alpha < \omega_1$, and is determined by its characteristic function $f_S : \alpha \rightarrow 2$. Since $|\alpha| < \omega_1$ in M , by ω_1 -closure in M of $\mathbb{P}$ we have $f_S \in M$. It follows $S \in M$.