Class info

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Assessment: 4-5 HWs (turned in on Gradescope)

Grading: standard cutoffs

Textbook: Kunen "Set Theory: An Introduction to Independence Proofs"
(the original book: 1st edition 1980
most recent edition 1992)

- will focus esp on ch. 3 + ch. 7
(some of ch. 4 + ch. 8)
- don't have to buy, will post all my notes on Canvas.

Class Overview

- Main topic is <u>forcing</u>: which is a technique for proving consistency and independence results over ZFC
- we saw a preview in 116b of forcing when we covered partial orders + MA.

Forcing and absoluteness: a cartoon

- Roughly, forcing is a method that
  1. <u>from</u> a countable transitive model M of ZFC (called the <u>ground model</u>)
  2. <u>constructs</u> a countable transitive model $M[G]$ of ZFC + [...]

where [...] can be e.g. CH
... or $\neg$CH
... or "$\exists$ a Suslin tree"
... or "MA + $2^\omega = \omega_{17}$"
etc.

- the G here is a <u>generic</u> <u>filter</u> for some $\underset{\text{poset}}{\mathbb{P}} \in M$.
- metamathematically, forcing gives what is called a "relative consistency proof"

i.e. 1 + 2 prove a statement of the form

"If ZFC is consistent (this is 1)
then so is ZFC + [...]" (this is 2)

or symbolically

$$\mathrm{Con}(ZFC) \Rightarrow \mathrm{Con}(ZFC + [\ldots])$$

- the point is: because of Gödel, we can't prove the consistency of ZFC from ZFC (i.e. can't prove $ZFC \Rightarrow \mathrm{Con}(ZFC)$)
- likewise can't prove things like $ZFC \Rightarrow \mathrm{Con}(ZFC + CH)$ or $ZFC \Rightarrow \mathrm{Con}(ZFC + \neg CH)$ etc.
- but forcing allows us to prove things like $\mathrm{Con}(ZFC) \Rightarrow \mathrm{Con}(ZFC + \neg CH)$
- slightly more precise to say this contrapositively: <u>if</u> $ZFC + \neg CH$ were inconsistent (i.e. proved a contradiction) then forcing gives us a way of pulling this back to a proof of contradiction just from ZFC

- Hence $ZFC + \neg CH$ must be consistent!
  ... <u>assuming</u> ZFC is.

- we'll (mostly) ignore the metamathematics and focus on model-theoretic conception above
- once we believe in countable models, the rest is math (not metamath)

<u>Absoluteness</u>

- let's examine 1 and 2 a little further (still in cartoon)

- imagine we work in the true universe of sets (this concept is a bit shaky but let's roll with it)
- call this universe V
- in V we have all familiar sets, e.g. the (actual) empty set $\emptyset$
the (actual) natural numbers ω
the (actual) reals $\mathbb{R}$ ... etc.

- We also have all the theorems from regular math, including 116a
- E.g. we have the Henkin construction, Gödel's completeness theorem, and the Lowenheim-Skolem theorem from 116a.
- Thus we know that if ZFC is consistent, then ZFC has a model

- Since language of set theory is ctbl (actually, finite) we know by LS theorem that in fact ZFC has a <u>countable</u> model
  (countable? how? more in a moment)

- i.e. there is a countable set N and a binary relation $E \subseteq N \times N$ s.t. $\langle N, E \rangle \models ZFC$

- $\langle N, E \rangle$ is a formal model: even in the objects $x \in N$ are sets, it need not be that E has anything to do w/ the actual set containment relation $\in$
- need only be that E as a relation on N satisfies ZFC.
- then, e.g., since $\langle N, E \rangle \models ZFC$ and ZFC proves there is a unique set w/ no elements, there must $\exists$ a (unique) $x \in N$ s.t. $\forall y \in N \; y \not E x$
- x is N's version of $\emptyset$
- but x may not be the actual $\emptyset$ (could really be anything! maybe $x = \{1, \pi, 5\}$)

But it turns out:

- one can construct (by taking a "Mostowski collapse") from the ctbl model $\langle N, E\rangle$ a <u>ctbl</u> <u>transitive</u> <u>model</u> (ctm) $\langle M, \in\rangle$
- i.e.

  M is a countable collection of sets
  $\in$ is the <u>actual</u> set membership relation
  M is <u>transitive</u> (i.e. $x \in M \Rightarrow x \subseteq M$)
  and $\langle M, \in\rangle \models ZFC$

↳ M is constructed naturally from N
e.g. we replace N's version of $\emptyset$ with the actual $\emptyset$, etc.
↳ indeed $\langle M, \in\rangle$ will be isomorphic to $\langle N, E\rangle$
- M is the copy of N w/ actual sets.

<u>Exploring the c.t.m. $\langle M, \in\rangle$</u>

- since $M \models ZFC$, $\exists!\, x \in M$ s.t. $\forall y \in M$ we have $y \notin x$
  red set membership now
- x is M's version of $\emptyset$
- we write $\emptyset^M$ for x
- but then: <u>since M is transitive</u> this x is the <u>actual</u> $\emptyset$
- <u>why</u>: if $\exists y \in x$, then $y \in M$ by transitivity and would have $M \models y \in x$ (since M is equipped w/ the actual $\in$)

- we write this as $\emptyset^M = \emptyset$
- this is an example of <u>absoluteness</u> between M and the actual universe V
- we say: the notion of the empty set is <u>absolute</u> between M and V

- similarly, since $M \models$ Pairing, $\exists! y \in M$ whose only element is $\emptyset$
- y is M's version of $1 = \{\emptyset\}$
- but again by transitivity of M must be that y is actually $\{\emptyset\}$
- we write $1^M = 1$

- similarly $2^M = 2$, $3^M = 3$, ... etc.

- since $M \models$ Infinity, M has its version of ω, call it ω^M
- can show (with a little more thought) $\omega^M = \omega$

- in fact a lot of things are absolute between M and V
- e.g. the concept of being an ordinal is absolute

  i.e. $M \models$ "x is an ordinal" iff x really is an ordinal

- If we let $o(M)$ denote the collection of ordinals that belong to M then by transitivity of M we have that $o(M)$ is an <u>initial segment</u> of ON (class of all ordinals)

- but now we run into <u>non</u>-absoluteness and the strangeness of M being countable
- since $M \models ZFC$ we have $M \models$ "$\exists$ an unctbl set" and in particular $M \models$ "there is a least unctbl ordinal"
- i.e. there is a least ordinal $\alpha \in M$ s.t. $\nexists$ a bijection $f: \alpha \to \omega$ <u>in M</u>.
- call this α ω_1^M
- by above, ω_1^M is an actual ordinal
- by transitivity we not only have $\omega_1^M \in M$ but $\omega_1^M \subseteq M$.
- since M is countable must be that ω_1^M is countable!
- i.e. M's version of ω_1 is a <u>countable</u> ordinal.
- so certainly $\omega_1^M \neq \omega_1$

- this example of non-absoluteness between M and V results from M being ctbl
- key point is: even though ω_1^M is actually ctbl, M cannot see this.

- this ex. shows: concept of "being ω_1" is not absolute.

- there are many similar "paradoxes" (really: failures of absoluteness)
- for example:
  - the concept of "being a subset of ω" is absolute
  - i.e. $M \models A \subseteq \omega$ iff A really is a subset of ω
  - since $M \models$ Powerset, $\exists$ a set $P(\omega)^M \in M$ that M believes is the powerset of ω
  - but in fact $P(\omega)^M$ only contains $A \subseteq \omega$ if $A \in M$ (i.e. $P(\omega)^M = P(\omega) \cap M$)
  - since $M \models ZFC$, have Cantor's theorem in M, so that $M \models P(\omega)^M$ is unctbl
  - i.e. $\nexists$ bij. $f : P(\omega)^M \to \omega$ in M
  - but from the outside we can see $P(\omega)^M$ is countable, since all of M is ctbl.

Forcing

- Vague question: can we exploit the countability of a ctm $M \models ZFC$ to produce new models of ZFC w/ targeted properties, like CH, $\neg$CH, $\square$ etc?

- Vague answer: Yes!
  ↳ it turns out a key point (proved in 116b) is that if $\mathbb{P}$ is a poset and $\langle D_n : n \in \omega \rangle$ is a countable collection of dense subsets of $\mathbb{P}$, then there is a filter $G \subseteq \mathbb{P}$ that intersects each one of them.

- More down to earth (but still vague) question: how might we pass from a ctm M to a larger ctm M'?
- e.g. suppose $A \subseteq \omega$ but $A \notin M$ (we know there are many such A, since M ctbl)
- can we find a ctm M' s.t. $A \in M'$?
- is there a "minimal" extension $M' \supseteq M$ s.t. $A \in M'$?
  ↳ this is where forcing comes in.

Let's recall an example from 116b.

<u>Ex</u> - Let $\mathbb{P} = \{p : p$ a finite partial function from ω to $2\}$
ordered $p \leq q$ if $p \supseteq q$.
- we think of conditions $p \in \mathbb{P}$ as approximating functions $f \in {}^{\omega}2$.
- for each $f \in {}^{\omega}2$ let
  $D_f = \{p \in \mathbb{P} : \exists n \in \mathrm{dom}(p) \;\; p(n) \neq f(n)\}$