<u>Def'n</u> if $\mathbb{P} = \langle P, \leq, \mathbb{1} \rangle$ is a p.o. define the $\mathbb{P}$-name $\check{x}$ recursively by:

$$\check{x} = \{\langle \check{y}, \mathbb{1} \rangle : y \in x\}$$

↳ since concepts involved in this recursive def'n are absolute, def'n of $\check{x}$ is absolute for transitive models of ZFC.

- hence if $x \in M$, $\check{x} \in M$ too

- some ex's:

$$\check{0} = 0 = \emptyset$$
$$\check{1} = \{0\}^{\vee} = \{\langle \check{0}, \mathbb{1} \rangle\} = \{\langle 0, \mathbb{1} \rangle\}$$
$$\check{2} = \{0,1\}^{\vee} = \{\langle \check{0}, \mathbb{1} \rangle, \langle \check{1}, \mathbb{1} \rangle\}$$
$$= \{\langle 0, \mathbb{1} \rangle, \langle \{0, \mathbb{1}\}, \mathbb{1} \rangle\}$$

etc.

- saw above: $\check{0}_G = 0$ and $\check{1}_G = 1$ for any $G$
  can check $\check{2}_G = 2$ " "
- More generally:

<u>Lemma</u>: if $M$ is a transitive model of ZFC, $\mathbb{P} \in M$ is a p.o., and $G$ a nonempty filter on $\mathbb{P}$, then:

a) $\forall x \in M$, $\check{x} \in M^{\mathbb{P}}$, and $\check{x}_G = x$
b) $M \subseteq M[G]$

<u>Pf</u>: a) already observed $\check{x} \in M^{\mathbb{P}}$ by absoluteness

$\check{x}_G = x$ follows by recursion using fact that $\check{x}_G = \{\check{y}_G : y \in x\}$

b) immediate from (a) ✓

- so: now we know $M \subseteq M[G]$
- to see $G \in M[G]$ need to find a name $\tau$ that always evaluates to $G$

<u>Def'n</u>: if $\mathbb{P}$ is a p.o., define
$$\Gamma = \{\langle \check{p}, p \rangle : p \in \mathbb{P}\}$$

* <u>Note</u> *: for remainder of lecture, assume $M$ is a ctm and $\mathbb{P} \in M$

- $\Gamma$ is going to be our name for $G$
- since (by def'n) $\Gamma$ is a $\mathbb{P}$-name involving only $\mathbb{P}$-names in $M$, we have $\Gamma \in M^{\mathbb{P}}$

<u>Lemma</u> $\Gamma_G = G$

<u>Pf</u>: $\Gamma_G = \{\check{p}_G : p \in G\} = \{p \in G\}$
$= G$ ✓

- so: $\Gamma$ is a canonical name for the generic filter $G$
- thus: while $\Gamma \in M$, (typically) $\Gamma_G = G$ is not

↳ $\Gamma$ is our first explicit example of a name that evaluates to something outside of $M$

- Lemma does give $G \in M[G]$, as promised

- Now: we eventually want to prove $M[G]$ is a ctm of ZFC
- is clearly countable!
- Also:

<u>Lemma</u>: $M[G]$ is transitive

<u>Pf</u>: more or less follows from def'n of names and $M[G]$:

- if $x \in M[G]$ then $x = \sigma_G$ for some $\sigma \in M^{\mathbb{P}}$
- $\sigma_G = \{\tau_G : \langle \tau, p \rangle \in \sigma$ for some $p \in G\}$
- so: every el't of $x$ is of the form $\tau_G$, hence is in $M[G]$
- i.e. $x \subseteq M[G]$ ✓

<u>Def'n</u> We write $o(M)$ for $ON \cap M$, i.e. $o(M)$ = set of ordinals in $M$.

<u>Lemma</u>: a) $\forall \tau \in M^{\mathbb{P}}$ $(\text{rank}(\tau_G) \leq \text{rank}(\tau))$
b) $o(M[G]) = o(M)$

<u>Pf</u>: - by induction on $\text{rank}(\tau)$
- clearly true if $\text{rank}(\tau) = 0$ (i.e. $\tau = \emptyset$) since then $\text{rank}(\tau_G) = 0$

- sps true for names $\sigma$ of rank $< \alpha$ and sps $\tau$ has rank $\alpha$
- then: $\text{rank}(\tau_G) = \text{rank}\{\sigma_G : \langle \sigma, p\rangle \in \tau$ and $p \in G\}$

$$= \sup\{\text{rank}(\sigma_G)+1 : \langle\sigma,p\rangle\in\tau, p\in G\}$$
$$\leq \sup\{\text{rank}(\sigma)+1 : \text{" "}\}$$
$$\overset{IH}{\leq} \sup\{\text{rank}(\langle\sigma,p\rangle)+1 : \langle\sigma,p\rangle\in\tau$$
$$= \text{rank}(\tau) \checkmark$$

(b) - We have $M \cap ON \subseteq M[G] \cap ON$ since $M \subseteq M[G]$

- But also $M[G] \cap ON \subseteq M \cap ON$:
  - fix $\alpha \in M[G] \cap ON$
  - then $\alpha = \tau_G$ for some $\tau \in M$
  - but $\alpha = \text{rank}(\alpha) \leq \text{rank}(\tau)$ by (a)
  - since $\text{rank}(\tau) \in M$ ($\text{rank}(x) \in M$ for <u>all</u> $x \in M$) and $\text{rank}(\tau)$ is an ordinal we have $\alpha \in M$ since $\alpha \leq \text{rank}(\tau)$ hence $\in \text{rank}(\tau)$ hence $\in M$ by transitivity. $\checkmark$

↳ conclusion of (b) worth repeating: even though $M[G]$ will (usually) contain objects that are not in $M$, it has the same ordinals as $M$!

- next want to verify some of the easier ZFC axioms in $M[G]$

- to verify Pairing, we'll need names for every pair of sets in $M[G]$

<u>Def'n</u> for names $\sigma, \tau \in M^{\mathbb{P}}$, define
ⓐ $up(\sigma,\tau) = \{\langle \sigma, \mathbb{1}\rangle, \langle \tau, \mathbb{1}\rangle\}$
ⓑ $op(\sigma,\tau) = up(up(\sigma,\sigma), up(\sigma,\tau))$

- note that $up(\sigma,\tau)$, $op(\sigma,\tau)$ are names (i.e. are in $M^{\mathbb{P}}$)

<u>Lemma</u>: ⓐ $up(\sigma,\tau)_G = \{\sigma_G, \tau_G\}$
ⓑ $op(\sigma,\tau)_G = \langle \sigma_G, \tau_G\rangle$

Pf: you try!

- the Lemma gives that Pairing holds in $M[G]$
- ... can get a few other axioms relatively easily also:

<u>Lemma</u>: Extensionality, Foundation, Pairing and Union are true in $M[G]$
(i.e. Extensionality$^{M[G]}$, ... etc. are precisely)

<u>Pf</u>: - we showed before Extensionality$^N$ holds for any transitive class $N$; $M[G]$ is transitive.

- $\text{Foundation}^{M[G]}$ written out is:

  $\forall x \in M[G] \ (\exists y \in x \ \forall z \in x \ (z \notin y))^{M[G]}$

  but this is absolute, hence

  $\text{Found}^{M[G]}$ equiv to:

  $\forall x \in M[G] \ (\exists y \in x \ \forall z \in x \ (z \notin y))$

  this is true $\forall x \in V$, hence def $\forall x \in M[G]$

  as $V \models$ Foundation

- Pairing is true by prev. lemma:

  if $x, y \in M[G]$ then $x = \sigma_G$, $y = \tau_G$

  for some $\sigma, \tau \in M^{\mathbb{P}}$

  so $\{x, y\} = \text{up}\langle \sigma, \tau \rangle_G \in M[G]$ too.

- recall we stated Union as

  $\forall a \exists b \ (\forall c \in a \ (\forall d \in c \ (d \in b)))$

  i.e. "$\forall a \exists b \ (\bigcup a \subseteq b)$" not "$\forall a \exists b \ (\bigcup a = b)$"

- let's verify this in $M[G]$
- fix $a \in M[G]$ and $\tau \in M^{\mathbb{P}}$ s.t. $\tau_G = a$.
- let (i.e. $\forall \langle \sigma, p \rangle \in \tau$

  $\pi = \bigcup \text{dom}(\tau)$ and $\forall \langle \sigma', p' \rangle \in \sigma$

  have $\langle \sigma', p' \rangle \in \pi$ )

- notice: $\pi$ is a name!
- let $b = \pi_G$
- now observe: if $d \in \bigcup a$ then $d = \sigma'_G$

  for some $\sigma'$ s.t. $\langle \sigma', p' \rangle \in \pi$ and $p' \in G$

- so $d \in \pi_G$ i.e. $d \in b \Rightarrow \bigcup a \subseteq b$ ✓