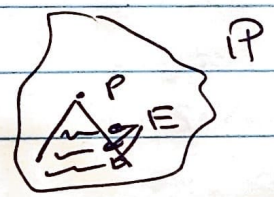
- will follow $U_a \in M[G]$ once we prove Comprehension holds in $M[G]$.

- before we dive further into forcing, need to generalize notion of "dense in $\mathbb{P}$" to "dense below a condition in $\mathbb{P}$"

<u>Def'n</u> given $p \in \mathbb{P}$ and $E \subseteq \mathbb{P}$, say E is <u>dense</u> <u>below</u> <u>p</u> if
$$\forall q \leq p \ \exists r \in E \ \ r \leq q$$



- idea is: if G is $\mathbb{P}$-generic and $p \in G$ then G intersects every dense-below-p set E ... let's prove!

<u>Lemma</u>: Sps M is a ctm of ZFC, $\mathbb{P} \in M$, $E \subseteq \mathbb{P}$ and $E \in M$. Let G be $\mathbb{P}$-generic over M. Then:

(a) Either $G \cap E \neq 0$, <u>or</u> $\exists q \in G \ \forall r \in E \ (q \perp r)$

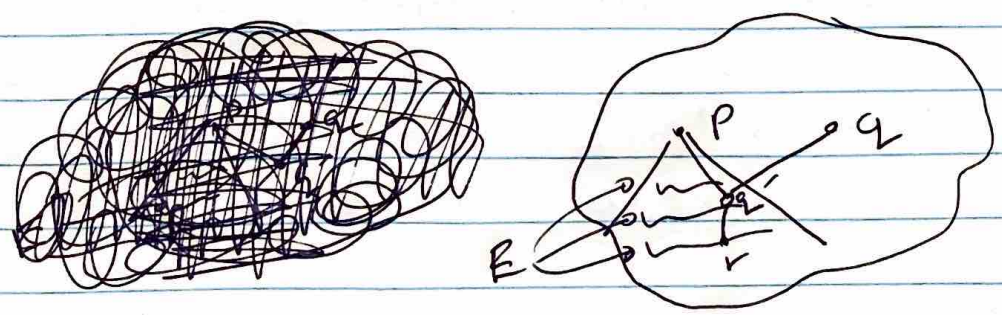
(b) if $p \in G$ and E is dense below p then $G \cap E \neq \emptyset$.

<u>Pf</u>: (a) let $D = \{p : \exists r \in E \ (p \leq r)\} \cup \{q : \forall r \in E \ (q \perp r)\}$

(= "downward closure of E" $\cup$ "orthogonal complement of E")

- <u>claim</u>: D is dense in $\mathbb{P}$
  - fix $q \in \mathbb{P}$
  - if $q \notin D$ then $q \not\leq$ any el't of E but q is compatible w/ some $r \in E$
  - i.e. can find $p \leq q, r$
  - but then $p \in D$ (since $p \leq r \in E$)
  - since $p \leq q$ too, D is dense ✓
- hence $G \cap D \neq \emptyset$ by genericity
- fix $q \in G \cap D$: must have <u>either</u> $q \leq$ some $r \in E$, in which case $r \in G$ since G is a filter and $G \cap E \neq \emptyset$ <u>or</u> $q \perp r$ $\forall r \in E$ ... this is (a)

(b) Sps E is dense below $p \in G$
- if we had $G \cap E = \emptyset$ then by (a) can find $q \in G$ s.t. $q \perp r$ $\forall r \in E$
- then, since G is downward directed can find $q' \leq q, p$ s.t. $q' \in G$
- then, since E is dense below p, can find $r \in E$ s.t. $r \leq q'$
- but then $r \leq q$, contradicting $r \perp q$ ✓



The forcing relation

- a key idea of forcing:
  M cannot reason directly about $M[G]$
  (since $G \notin M$ .. usually)
  but: M can reason conditionally about $M[G]$ — conditionally on knowledge of G.

Ex Sps M is a ctm and $\mathbb{P} = \{p :$ p a finite partial function from ω to $2\}$
ordered: $p \leq q$ if $p \supseteq q$
$\mathbb{1}$ = empty function (max'l in $\mathbb{P}$)

- notice: $\mathbb{P} \in M$
  why: $\mathbb{P} \subseteq M$ (el'ts of $\mathbb{P}$ are finite functions and M contains all of its finite sets)
  and "p is a finite partial function from ω to 2" is absolute
  hence $\mathbb{P}^M = \mathbb{P}$.

- if G is a filter on $\mathbb{P}$, then $f_G = \bigcup G$ is a function (the partial functions in G are compatible)
- in general only know $\mathrm{dom}(f_G) \subseteq \omega$
  (i.e. f_G may be partial)

- for every n, let $D_n = \{p : n \in \mathrm{dom}(p)\}$
- each D_n is dense in $\mathbb{P}$ (why?)
- moreover: each $D_n \in M$ (def'n of D_n is absolute, i.e. $D_n^M = D_n$)
- thus if G generic, $G \cap D_n \neq \emptyset$ $\forall n \in \omega$
  $\Rightarrow \mathrm{dom}(f_G) = \omega$, i.e. $f_G : \omega \to 2$ is <u>total</u>

- claim $f_G \in M[G]$ (intuitive if we believe $M[G]$ consists of objects constructible from G)
- if we take $M[G] \models ZFC$ as granted, this is immediate:
  $(\bigcup G)^M \in M[G]$ by Union
  and $(\bigcup G)^M = \bigcup G = f_G$ by absoluteness of "$\bigcup G$"

- but we can also show $f_G \in M[G]$ directly by writing down a name Φ that must evaluate to f_G
- let $\Phi := \{\langle \langle n,m \rangle^\vee, p \rangle : \langle n,m \rangle \in p\}$
  (i.e. $n \in \mathrm{dom}(p) \wedge p(n) = m$)
- observe:
  $$\Phi_G = \{\langle n,m \rangle : \exists p \in G \;\; p(n) = m\}$$
  $$= \bigcup G = f_G$$
- hence $f_G \in M[G]$

- not hard to check $\mathbb{P}$ satisfies splitting condition, i.e. $\forall p \in \mathbb{P}\ \exists q, r \leq p\ \ q \perp r$

  take $q, r \leq p$ to be ext'ns of p s.t. for some $m \notin \mathrm{dom}(p)$, $q(m) = 0$, $r(m) = 1$

$\Rightarrow$ if G $\mathbb{P}$-generic over M then $G \notin M$.

- follows: $f_G \notin M$ either:

  if it were could define (in M) $\{p : p \subseteq f_G\}$ $= G$. But then would have $G \in M$.

- from p.o.v. of M, f_G is a new function (a "transcendental function")
- M can't see G or f_G but does have names Γ and Φ for them

- M can't know f_G, but can reason as we did conditionally about f_G, e.g. can reason

  "f_G will be a function from ω to 2"

- M can't compute e.g. $f_G(0)$, but can reason conditionally:

  "if $p = \{\langle 0,0\rangle\} \in G$ then $f_G(0) = 0$"

  "if $q = \{\langle 0,1\rangle\} \in G$ then $f_G(0) = 1$"

as we did

- More formally, M can construct what's called a <u>forcing language</u>

- this lang. uses names to write statements about what might be true in $M[G]$
- e.g. let ψ denote

name for f_G → $\underline{\Phi}(\check{0}) = \check{1}$

↳ note this fmla is not written in lang. of set theory but in a similar lang that allows use of names (we call this the forcing language)

↳ ψ intuitively asserts $f_G(0) = 1$, but it uses the names for these objects to assert this (M has access to names for objects in $M[G]$, not to the objects themselves)

- M won't know in general whether a given fmla ψ written in the forcing lang is true in $M[G]$
- however it can know conditionally whether ψ is true depending on which conditions belong to G

<u>We write</u>: $p \Vdash \psi$ ("p forces ψ")
to mean ψ is true in $M[G]$ whenever $p \in G$.

- e.g. can express the above observations as:

"$f_G(0) = 0$"

$\{\langle 0,0\rangle\} \Vdash \Phi(\check{0}) = \check{0}$

also: $\{\langle 0,1\rangle\} \Vdash \Phi(\check{0}) = \check{1}$ ← "$f_G(0) = 1$"

also: $\mathbb{1} \Vdash \Phi$ is a function from $\check{\omega}$ to $\check{2}$
(since this is true in $M[G]$ for every G)

→ once we formalize all this, will be able to show two key facts about the forcing relation $\Vdash$

<u>Fact 1</u> For a given fmla ψ (written in the forcing language) and $p \in \mathbb{P}$, it can be decided <u>in M</u> whether $p \Vdash \psi$

- this generalizes our "conditional reasoning" examples above

- to state second fact, observe that if there is $p \in G$ s.t. $p \Vdash \psi$, then (by def'n of $\Vdash$) ψ is true in $M[G]$
- we'll show converse is also true!

<u>Fact 2</u>: if G is $\mathbb{P}$-generic over M and ψ is true in $M[G]$ then $\exists p \in G$ s.t. $p \Vdash \psi$.

- can view Fact 2 as evidence that generic filters over posets $\mathbb{P} \in M$ are the "right" way to minimally add information to M.

- once we establish (formal versions of) Facts 1+2 (or... at least state..) will be able to prove a lot of independence results

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- let's record a formal def'n of the forcing relation $\Vdash$:

<u>Def'n</u> Sps $\varphi(x_1, \ldots, x_n)$ is a fmla w/ free variables among $x_1, \ldots, x_n$.
Sps M is a ctm of ZFC, $\mathbb{P} \in M$, and $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$ are names in M.
Given $p \in \mathbb{P}$ we define:
$p \Vdash \varphi(\tau_1, \ldots, \tau_n)$ iff
for every $\mathbb{P}$-generic filter G with $p \in G$, we have $\varphi^{M[G]}(\tau_{1G}, \ldots, \tau_{nG})$

↳ So: $p \Vdash \varphi(\tau_1, \ldots, \tau_n)$ iff φ is true of the realized elements $\tau_{1G}, \ldots, \tau_{nG}$ in any extension $M[G]$ by a filter G containing p.

- as we noted above: is a trivial consequence of def'n of $\Vdash$ that if there is $p \in G$ s.t. $p \Vdash \varphi$ then φ is true in $M[G]$
- following theorem is fundamental theorem of forcing — says converse is true.

<u>Theorem</u> (The Forcing Theorem) Sps M is a ctm, $\mathbb{P} \in M$, G is $\mathbb{P}$-generic$/M$. Sps $\varphi(x_1, \ldots, x_n)$ is a fmla and $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$ are names.

<u>Then</u>: $\varphi(\tau_{1_G}, \ldots, \tau_{n_G})^{M[G]}$ iff $\exists p \in G \;\; p \Vdash \varphi(\tau_1, \ldots, \tau_n)$

↳ this is formally written out version of Fact 2 above.
- says "everything true in $M[G]$ is forced to be true by some $p \in G$"

- we will delay proof for now (which is involved) and dive into applications
- <u>using</u> the Forcing Theorem often requires a lot of ingenuity, but not so much knowledge of its proof.
- akin to the Fund. Thm. of Calc. in this way.