- what about Fact 1?
- since "$p \Vdash \varphi$" is defined by quantifying over all G that are $\mathbb{P}$-generic $/M$, doesn't make sense to consider $(p \Vdash \varphi)^M$

  meaningless, as $G \notin M$
- so how can we formalize "It can be decided in M whether $p \Vdash \varphi$"?

<u>Theorem</u> there is a relation $\Vdash^*$ defined by an absolute recursion s.t. (under same hypotheses as theorem above) for all $p \in \mathbb{P}$:

$$(p \Vdash^* \varphi(\tau_1, \ldots, \tau_n))^M \text{ iff } p \Vdash \varphi(\tau_1, \ldots, \tau_n)$$

- The last theorem we need is the one promised for a long time:

<u>Theorem</u> ZFC is true in $M[G]$

Pf: also deferred for now.

- Before we take the Forcing Theorem out for a spin, let's verify a simple fact we'll use repeatedly

<u>Lemma</u> Sps $p \in \mathbb{P}$ and $p \Vdash \varphi(\tau_1, \ldots, \tau_n)$
Then $\forall q \leq p$, $q \Vdash \varphi(\tau_1, \ldots, \tau_n)$

Pf: - if G is generic and $q \in G$ then $p \in G$ too (since G a filter)
- hence $\varphi(\tau_{1_G}, \ldots, \tau_{n_G})$ is true in $M[G]$ (since p forces it)
- hence $q \Vdash \varphi(\tau_1, \ldots, \tau_n)$ etc. ✓

Forcing the continuum to be large

- First goal w/ forcing: show that both ZFC + CH and ZFC + $\neg$CH are consistent (relative to consistency of ZFC ... i.e. assuming $\exists$ a ctm)
- i.e. answer to Hilbert's 1st problem "is CH true?" is: could go either way!
- we'll do $\neg$CH first (i.e. $|\mathcal{P}(\omega)| > \omega_1$ is consistent)

Forcing $\neg$CH:

- here's basic idea: we can view $\mathbb{P} = \{p : p \text{ a fin. part'l function from } \omega \text{ to } 2\}$ as the forcing poset to "add a single subset of ω"
  - what's this mean?
  - if G is $\mathbb{P}$-generic and $f_G = \bigcup G$ is assoc. function then the subset in question is $A_G = \{n \in \omega : f_G(n) = 1\}$ (f_G is char. function for A_G)

- observe $A_G \in M[G]$ but $A_G \notin M$
  (why: can define A_G from f_G and vice versa)
- so A_G is a "new subset of ω" in $M[G]$

- idea for making $|\mathcal{P}(\omega)| > \omega_1$ in $M[G]$ is to add <u>lots</u> of subsets of ω all at once.
- trick is to do this in a way that doesn't change values of cardinals in $M[G]$
- we'll use partial functions again

<u>Def'n</u> Given sets I, J we define the poset:
$$Fn(I,J) := \{p : p \text{ is a finite partial function from } I \text{ to } J\}$$

- ordering as usual: $p \leq q$ iff $p \supseteq q$
  $\mathbb{1}$ is the empty function

- for remainder, fix a ctm M.
- since "finite" (+ everything else in def'n of $Fn(I,J)$) is absolute, if $I, J \in M$ then $Fn(I,J)^M = Fn(I,J)$ so $Fn(I,J) \in M$
- e.g. our $\mathbb{P}$ from before is $Fn(\omega, 2)$
- if G is a filter for $Fn(I,J)$ then $f_G = \bigcup G$ is a function w/ $dom(f_G) \subseteq I$ and $ran(f_G) \subseteq J$.

<u>Prop'n</u> Sps $I, J \in M$ and I is infinite. Sps G is generic for $Fn(I,J)$ over M. Then $f_G = \bigcup G$ is total (i.e. $dom(f_G) = I$) and onto J (i.e. $ran(f_G) = J$)

<u>Pf</u>: ① totality: for each $x \in I$, $D_x = \{p \in Fn(I,J) : x \in dom(p)\}$ is dense (why? need I infinite here)

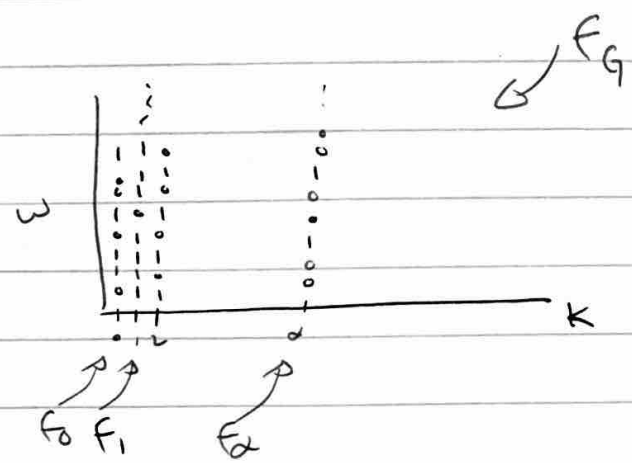
② surjectivity: for each $y \in J$, $D_y = \{p \in Fn(I,J) : y \in dom(p)\}$ is dense (why?) ✓

- As we've seen: since M is ctbl, "being a cardinal" is not absolute between M and V (e.g. ω_1^M is a ctbl ordinal)
- we can now show "being a cardinal" is not even absolute between M and $M[G]$.
- fix $\kappa \in M$ s.t. (κ is an unctbl cardinal)^M (e.g. could take $\kappa = \omega_1^M$)
- let $\mathbb{P} = Fn(\omega, \kappa)$ and let G be $\mathbb{P}$-generic / M.
- then in $M[G]$, $f_G = \bigcup G$ is a surjection from ω onto κ
- so, in $M[G]$, κ still an ordinal (which is abs) but now a ctbl ordinal (hence no longer a cardinal)
- of course <u>all</u> ordinals in M are <u>actually</u> ctbl ordinals ... we've now added a function to $M[G]$ that can see that a particular

cardinal κ that M thought was an unctbl cardinal is actually ctbl.

- we say: $\mathbb{P}$ <u>collapses</u> κ
- if $\kappa = \omega_1^M$ then in particular this gives $\omega_1^{M[G]} > \omega_1^M$.

- we can also use a poset of the form $Fn(I, J)$ to make 2^ω large.
- let κ be an unctbl cardinal in M (i.e. $\kappa \in M$ and (κ is unctbl card'l)^M)
- let $\mathbb{P} = Fn(\kappa \times \omega, 2)$
- fix G $\mathbb{P}$-generic$/M$, then $f_G = \bigcup G$ is a function from $\kappa \times \omega \to 2$
- can think of f_G as being a sequence of κ-many functions from $\omega \to 2$
- i.e. for each $\alpha \in \kappa$, can define $f_\alpha : \omega \to 2$ by $f_\alpha(n) = f_G(\alpha, n)$

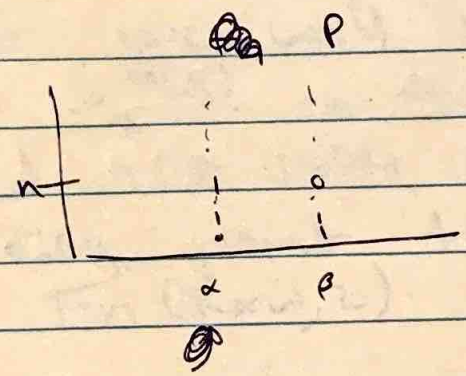


- observe each $f_\alpha \in M[G]$ (definable in absolute way from f_G)

- in fact the set $\{f_\alpha : \alpha < \kappa\} \in M[G]$ (definable in abs. way from f_G)
- hence in $M[G]$ 2^ω $(= |{}^\omega 2|)$ is $\geq$ $|\{f_\alpha : \alpha \in \kappa\}|$ since they set ω $\subseteq {}^\omega 2$

- we can also show the f_α are pairwise distinct:

  <u>why</u>: if $\alpha, \beta < \kappa$ and $\alpha \neq \beta$ define
  $D_{\alpha\beta} = \{p \in \mathbb{P} : \exists n \in \omega$ s.t. $\langle \alpha, n\rangle, \langle \beta, n\rangle \in \mathrm{dom}(p)$
  $\wedge\ p(\langle \alpha, n\rangle) = 0 \wedge p(\langle \beta, n\rangle) = 1\}$



then $D_{\alpha\beta}$ is dense (why?)
$\Rightarrow \exists p \in G \cap D_{\alpha\beta} \Rightarrow f_\alpha(n) \neq f_\beta(n)$ for some $n \in \omega$

- all to say: $\exists$ a κ-sequence $\{f_\alpha : \alpha \in \kappa\}$ from ω to 2 in $M[G]$
- we're proved

<u>Prop'n</u> Given $\kappa \in M[G]$, if G is $\mathrm{Fn}(\kappa \times \omega, 2)$ generic / M then $(2^\omega \geq |\kappa|)^{M[G]}$

- so let $K = \omega_2^M$
- then in $M[G]$ we have $2^\omega \geq K$
- so: $2^\omega \geq \omega_2$ in $M[G]$, right? $\exists$
  i.e. $\neg CH$ holds in $M[G]$ i.e. $(2^\omega \geq K)^{M[G]}$

↳ not quite! we've only shown that $2^\omega \geq \omega_2^M$ in $M[G]$
- it may be that ω_2^M is $< \omega_2^{M[G]}$
  (we saw this can happen w/ c.s. ω_1 above)

- however: we will show that in fact $\omega_2^M = \omega_2^{M[G]}$ in this case, so that indeed $\neg CH$ holds in $M[G]$
- the key point turns out to be that $Fn(K \times \omega, 2)$ has the ccc in M

<u>Lemma</u> If I is arbitrary and J is countable then $Fn(I,J)$ has the ccc
(recall: $\mathbb{P}$ has the ccc if $\nexists$ unctbl antichains in $\mathbb{P}$)

Pf: you showed this on last HW of 116b.

- <u>note</u>: lemma is asserted in V (i.e. this is a fact about $Fn(I,J)$ when J ctbl – doesn't depend on any M)