More cardinal manipulation: Forcing w/ functions of larger cardinality

- Another Q: can we manipulate 2^λ (i.e. $|P(\lambda)|$) for $\lambda > \omega$?
- Yes! Same approach works:
  - e.g. if we want to get $2^{\omega_1} = \omega_{17}$
  - can force with $\mathbb{P} = Fn(\omega_{17}^M \times \omega_1^M, 2)$

(i.e. the poset to "add ω_{17}-many subsets of ω_1")
and show that in $M[G]$

$$2^{\omega_1} \geq \omega_{17} \text{ and (if } GCH \text{ holds in } M)$$
$$2^{\omega_1} \leq \omega_{17} \quad \text{(by nice name lemma)}$$
$$\Rightarrow 2^{\omega_1} = \omega_{17} \text{ in } M[G]$$

- interesting point: in $M[G]$ we'll also have $2^\omega = \omega_{17}$ (i.e. $(2^\omega = 2^{\omega_1} = \omega_{17})^{M[G]}$)
- and in fact, in $M[G]$ (assuming GCH in M)

$$2^\omega = 2^{\omega_1} = 2^{\omega_2} = \cdots = 2^{\omega_{16}} = \omega_{17}$$

Pf: maybe HW.

- so 2^ω, 2^{ω_1} have been blown up in $M[G]$ ... but to the same cardinal
- this turns out to be unavoidable when forcing w/ $Fn(\kappa \times \omega^M, 2)$
  - ↳ what if we want to blow up 2^ω, 2^{ω_1} at same time, but keep them separate?
- e.g. can we get consistency of $2^\omega = \omega_5$ and $2^{\omega_1} = \omega_{17}$?

- <u>Idea</u>: forcing w/ functions w/ larger domains will give us such flexibility
- ironically will also give us power to force CH.

<u>Def'n</u> For λ an infinite cardinal, let
$Fn(I,J,\lambda) := \{p : p$ is a partial function from I to J w/ $|dom(p)| < \lambda\}$

↳ order by $p \leq q$ if $p \supseteq q$ (as usual)
↳ $\mathbb{1}$ is empty function (as usual)

- in this notation: $Fn(I,J)$ is $Fn(I,J,\omega)$
- $Fn(I,J,\omega_1)$ is all partial $p : I \to J$ w/ at most ctbl domain.

<u>Note</u>: for $\lambda > \omega$, $Fn(I,J,\lambda)$ is <u>not</u> absolute for a ctm M.
↳ we'll always be forcing w/ $Fn(I,J,\lambda)^M$ (i.e. M's version of $Fn(I,J,\lambda)$) where $I,J,\lambda \in M$ and (λ is a cardinal)^M
↳ from outside, conditions $p \in Fn(I,J,\lambda)^M$ always have at most ctbl domain
↳ but if $(\lambda > \omega_1)^M$ they could have unctbl domain in M.
↳ even if $\lambda = \omega_1^M$, so that $p \in Fn(I,J,\omega_1)^M$ have ctbl domain both inside + outside M, there will be ctbl partial $f : I \to J$ w/ $f \notin M$.

<u>Lemma</u> Sps $I, J, \lambda \in M$, $(\lambda$ is a cardinal$)^M$, $J \neq 0$, and $(|I| \geq \lambda)^M$. Then if G is $Fn(I, J, \lambda)^M$-generic$/M$, then $f_G = \bigcup G$ is a function from I onto J.

<u>Pf</u>: - same as w/ $Fn(I,J)$
- just need $|I| \geq \lambda$ in M so we can always extend $p \in Fn(I,J,\lambda)^M$ to conditions q w/ a given $i \in I$ in their domain, or $j \in J$ in their range. ✓

<u>Lemma</u> Sps $(\lambda$ is a cardinal$)^M$ and $K \in M$. If G is $Fn(K \times \lambda, 2, \lambda)^M$-generic$/M$, then $(2^{|K|} \geq |K|)^{M[G]}$

<u>Pf</u>: - same as w/ $Fn(K \times \lambda, 2)$:
- can split $f_G = \bigcup G$ into $|K|$-many distinct functions $f_\alpha : \lambda \to 2$
- writing $2^{|K|}$ here since a priori in $M[G]$ λ may no longer be a cardinal (we'll show it usually is) ✓

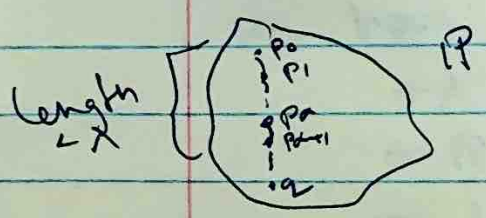
↳ to continue analogy w/ $Fn(I,J)$ would next want to show $Fn(I,J,\lambda)$ preserves cardinals
- turns out: not true in general!
- will often be true depending on cardinal arithmetic in M.

- key point is: new arg. splits
  (i) to show card'es $> \lambda$ are preserved, need an analogue of ccc-ness for $Fn(I,J,\lambda)$ (+ some cardinal arithmetic in M)
  (ii) to show cardinals $\leq \lambda$ are preserved need a new idea called <u>λ-closure</u> ... we'll introduce this first.

<u>Def'n</u> Spse $\mathbb{P} \in M$ and $\kappa \in$ is an infinite cardinal$)^M$. We say $\mathbb{P}$ <u>preserves cardinals $\geq \kappa$</u> (resp. $\leq \kappa$) if whenever G is $\mathbb{P}$-generic$/M$, $\beta \in o(M)$, and $\beta \geq \kappa$ (resp. $\beta \leq \kappa$) we have
$$(\beta \text{ is a cardinal})^M \text{ iff } (\beta \text{ is a cardinal})^{M[G]}$$

↳ showing $Fn(I,J,\lambda)$ preserves cards $\leq \lambda$ requires a new def'n

<u>Def'n</u> Spse λ is a cardinal. We say $\mathbb{P}$ is <u>λ-closed</u> if whenever $\gamma < \lambda$ and $\{p_\alpha : \alpha < \gamma\}$ is a decreasing γ-length sequence of elements of $\mathbb{P}$ (i.e. $p_0 \geq p_1 \geq \dots \geq p_\alpha \geq \dots$) then:
$$\exists q \in \mathbb{P} \ \forall \alpha < \gamma \quad q \leq p_\alpha$$



- if $\lambda = \omega_1$, then λ-closure means that for any <u>ctbl</u> decreasing sequence p_i $\exists q \in \mathbb{P}$ s.t. $q \leq p_i$ $\forall i$.
- sometimes ω_1-closure is called <u>ctbl closure</u> for this reason

<u>Lemma</u> if λ <u>is regular</u> then $Fn(I, J, \lambda)$ is λ-closed.

Pf: - if $p_0 \geq p_1 \geq \cdots \geq p_\alpha \geq \cdots$ is a $<\lambda$-length sequence in $Fn(I, J, \lambda)$ then $q = \bigcup p_\alpha$ lies below all p_α
- note: $|dom(q)| < \lambda$ by regularity of λ, so $q \in Fn(I, J, \lambda)$ ✓

↳ not hard to see: if λ is singular then $Fn(I, J, \lambda)$ is <u>not</u> λ closed (why?)

- following is the key theorem:
- says if $(\mathbb{P}$ is λ-closed$)^M$ then we don't add any functions w/ small domain as we pass to $M[G]$.

<u>Theorem</u> Sps $\mathbb{P} \in M$, $A, B \in M$, and in M we have:

- λ is a cardinal
- $\mathbb{P}$ is λ-closed
- $|A| < \lambda$

Sps G is $\mathbb{P}$-generic / M. Then: if $f: A \to B$ and $f \in M[G]$, then in fact $f \in M$.

Pf: – fix $f: A \to B$ s.t. $f \in M[G]$

– sps toward contradiction $f \notin M$, i.e. $f \notin ({}^A B)^M$ (which is ${}^A B \cap M$ by absoluteness)

– let $K = ({}^A B)^M$ (so $K \in M$)

– fix a name $\tau \in M^{\mathbb{P}}$ s.t. $\tau_G = f$

– since $f: A \to B$ and $f \notin M$, must be a condition $p_0 \in G$ s.t.

$$\underbrace{p_0 \Vdash \tau \text{ is a function from } \check{A} \text{ to } \check{B} \text{ and } \tau \notin \check{K}}_{\text{we derive a contradiction from this}}$$

– since $|A| < \lambda$ in M, there is $\alpha < \lambda$ and an enumeration $\{a_\gamma : \gamma < \alpha\} \in M$ of A in order type α.

– using AC + recursion in M, we find sequences $\{p_\gamma : \gamma < \alpha\}$ of el'ts of $\mathbb{P}$ and $\{b_\gamma : \gamma < \alpha\}$ of el'ts of B

s.t.

① $p_\eta \leq p_\gamma$ for $\eta > \gamma$ (i.e. sequence is decreasing)

② $p_{\gamma+1} \Vdash \tau(\check{a}_\gamma) = \check{b}_\gamma$

(i.e. $p_{\gamma+1}$ decides value of $\tau(a_\gamma)$ to be b_γ)