- e.g. if $\tau_1 = \emptyset$ and $\tau_2 = \emptyset$ (max rank$(\tau_1,\tau_2) = 0$)
then every $p \Vdash \tau_1 = \tau_2$
- if $\tau_1 = \{\langle \emptyset, s\rangle\}$, $\tau_2 = \emptyset$ (max rank $= 1$)
when does $p \Vdash \tau_1 = \tau_2$?
  ↳ every $G \ni p$ must have $s \notin G$
  ↳ this equiv to $p \perp s$ (why?)

- but verifying if $p \Vdash \tau_1 = \tau_2$ gets more complicated as the ranks of τ_1, τ_2 increase.
- we'll define $p \Vdash^* \tau_1 = \tau_2$ recursively (on name rank)
- to motivate, consider $\tau_1 = \{\langle \pi_1, s\rangle\}$
$\tau_2 = \{\langle \pi_2, s\rangle\}$
  ↳ when in this case does $p \Vdash \tau_1 = \tau_2$?
    - if $p \perp s$ then $p \Vdash \tau_1 = \tau_2$ (since $G \ni p \Rightarrow \tau_{1G} = \tau_{2G} = \emptyset$)
    - if $p \leq s$, then if $p \in G$ we have $\tau_{1G} = \{\pi_{1G}\}$, $\tau_{2G} = \{\pi_{2G}\}$
    - so $p \Vdash \tau_1 = \tau_2$ iff $p \Vdash \pi_1 = \pi_2$

- this tips us off to a recursion: if we can already decide if $p \Vdash \pi_1 = \pi_2$ (these names have lower rank than τ_1, τ_2) then can decide if $p \Vdash \tau_1 = \tau_2$

- our def'n for $p \Vdash^* \tau_1 = \tau_2$ will look somewhat complicated

- point is: will depend recursively on whether $q \Vdash^* \pi_1 = \pi_2$ for certain $q \in \mathbb{P}$, $\pi_1, \pi_2 \in \mathrm{dom}(\tau_1) \cup \mathrm{dom}(\tau_2)$

- <u>note</u>: we make the following def'n in V, but in args will always be relativizing to some etm M

<u>Def'n</u> Spse $\mathbb{P}$ is a poset. For a condition $p \in \mathbb{P}$, fmla $\varphi(x_1, \ldots, x_n)$, and names $\tau_1, \ldots, \tau_n \in V^{\mathbb{P}}$, we define $p \Vdash^* \varphi(\tau_1, \ldots, \tau_n)$ recursively (on names rank), by induction on construction of φ.

a) $p \Vdash^* \tau_1 = \tau_2$ iff

(i) $\forall \langle \pi_1, s_1 \rangle \in \tau_1$

$\{q \leq p : q \leq s_1 \Rightarrow \exists \langle \pi_2, s_2 \rangle \in \tau_2 \; (q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$

is dense below p, <u>and</u>

think "$\tau_1 \subseteq \tau_2$"

(ii) $\forall \langle \pi_2, s_2 \rangle \in \tau_2$

$\{q \leq p : q \leq s_2 \Rightarrow \exists \langle \pi_1, s_1 \rangle \in \tau_2 \; (q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2)\}$

is dense below p

b) $p \Vdash^* \tau_1 \in \tau_2$ iff

$\{q : \exists \langle \pi, s \rangle \in \tau_2 \; (q \leq s \wedge q \Vdash^* \pi = \tau_1)\}$

is dense below p

c) $p \Vdash^* (\varphi(\tau_1, \dots, \tau_n) \wedge \psi(\tau_1, \dots, \tau_n))$
iff $p \Vdash^* \varphi(\tau_1, \dots, \tau_n)$ and $p \Vdash^* \psi(\tau_1, \dots, \tau_n)$

d) $p \Vdash^* \neg \varphi(\tau_1, \dots, \tau_n)$ iff
there is no $q \leq p$ s.t. $q \Vdash^* \varphi(\tau_1, \dots, \tau_n)$

e) $p \Vdash^* \exists x\, \varphi(x, \tau_1, \dots, \tau_n)$ iff
$\{r : \exists \sigma \in V^{\mathbb{P}} (r \Vdash^* \varphi(\sigma, \tau_1, \dots, \tau_n))\}$
is dense below p.

↳ can think of the clauses "this set is dense below p" in these def'ns as stand-ins for "every generic $G \ni p$ intersects this set"

<u>Explaining the def'n:</u>

- def'n is in order:
  ① we first define $p \Vdash^* \tau_1 = \tau_2$ by recursion on names
  - more formally, our recursive def'n is: if $\forall$ names π_1, π_2 of rank $< \alpha$ we've defined $q \Vdash^* \pi_1 = \pi_2$ $(\forall q \in \mathbb{P})$, then for names τ_1, τ_2 of rank $\leq \alpha$ we define $p \Vdash^* \tau_1 = \tau_2$ as above $(\forall p \in \mathbb{P}$
  - works because def'n of $p \Vdash^* \tau_1 = \tau_2$ depends only on knowing $q \Vdash^* \pi_1 = \pi_2$ for $\pi_1 \in \mathrm{dom}(\tau_1)$, $\pi_2 \in \mathrm{dom}(\tau_2)$ (i.e. names of lower rank)

② having defined $p \Vdash^* \tau_1 = \tau_2$ $\forall p \forall \tau_1, \tau_2$ then we define $p \Vdash^* \tau_1 \in \tau_2$ $\forall p, \tau_1, \tau_2$
– works since def'n depends only on rel'n $p \Vdash^* \tau_1 = \tau_2$

③ then we define, by induction on construction of fmlas φ, for every fmla $\varphi(x_1, \ldots, x_n)$ and $\forall \tau_1, \ldots, \tau_n$ names, the relation $p \Vdash^* \varphi(\tau_1, \ldots, \tau_n)$ $(\forall p \in \mathbb{P})$

– tho we've defined in V, we will always be concerned w/ relativization of $\Vdash^*$ to some ctm M, i.e. the relation $(p \Vdash^* \varphi(\tau_1, \ldots \tau_n))^M$ for some $\mathbb{P} \in M$ and $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$.

– can convince yrself: def'ns of $p \Vdash^* \tau_1 = \tau_2$ and $p \Vdash^* \tau_1 \in \tau_2$ absolute for ctm's.
– def'n of $p \Vdash^* \varphi$ not absolute in general: in ⓔ we quantify over all $\sigma \in V^{\mathbb{P}}$; in relativized version this becomes over all $\sigma \in M^{\mathbb{P}}$.

– it's a triviality that $\Vdash^*$ is definable in V (we've just defined it!)
– hence $\Vdash^{*M}$ is definable in a given M.
– WTS: $\Vdash^{*M}$ captures $\Vdash$ (then $\Vdash$ is definable in M – this is Fact 1)

- i.e. we want to show
  (i) if $p \Vdash^* \varphi$ holds in M, φ holds in $M[G]$
  (ii) if φ holds in $M[G]$, $\exists p \in G$ $p \Vdash^* \varphi$.

- proof will be induction using def'n of $\Vdash^*$
- will need following lemma (relativized to M)

<u>Lemma</u> for $p \in \mathbb{P}$, a fmla $\varphi(x_1, \ldots, x_n)$, and names $\tau_1, \ldots, \tau_n \in V^{\mathbb{P}}$, TFAE:
① $p \Vdash^* \varphi(\tau_1, \ldots, \tau_n)$
② $\forall r \leq p$, $r \Vdash^* \varphi(\tau_1, \ldots, \tau_n)$
③ $\{r \in \mathbb{P} : r \Vdash^* \varphi(\tau_1, \ldots, \tau_n)\}$ is dense below p.

<u>Pf</u>: ② $\Rightarrow$ ③ is trivial
- for ① $\Rightarrow$ ②, ③ $\Rightarrow$ ① we first assume φ is atomic, i.e. φ is $\tau_1 = \tau_2$ or $\tau_1 \in \tau_2$.

① $\Rightarrow$ ②: follows from def'n of $p \Vdash^* \tau_1 = \tau_2$ and $p \Vdash^* \tau_1 \in \tau_2$
<u>Why</u>: - if D is any set dense below p, and $r \leq p$, then D is dense below r.
- def'ns of $p \Vdash^* \tau_1 = \tau_2$ $p \Vdash^* \tau_1 \in \tau_2$ depend on a set being dense below p.

③ $\Rightarrow$ ①: - follows (why?) from fact that: for $D \subseteq \mathbb{P}$, if $\{r : D$ is dense below $r\}$ is dense below p, then

D is dense below p. ✓

- now we prove ① ⇒ ② and ③ ⇒ ① for general φ by induction on construction of φ.

- if ① ⇒ ② and ③ ⇒ ① both true for φ, ψ then both true for $\varphi \wedge \psi$ (immediate)
  (only place we need inductive hyp.)

- <u>for $\neg\varphi$:</u>
  - <u>① ⇒ ②</u>: $p \Vdash^* \neg\varphi$ iff
    $\nexists q \leq p$ s.t. $q \Vdash^* \varphi$ iff
    $\forall r \leq p \ \nexists q \leq r$ s.t. $q \Vdash^* \varphi$ iff
    $\forall r \leq p \quad r \Vdash^* \neg\varphi$ ✓
  - <u>③ ⇒ ①</u>: - sps densely many $r \leq p$, $r \Vdash^* \neg\varphi$
    - if we could find $q \leq p$ s.t. $q \Vdash^* \varphi$ then could find $r \leq q$ (by density) s.t. $r \Vdash^* \neg\varphi$, contradicting $q \Vdash^* \varphi$
    - hence no such q
    - hence $p \Vdash^* \neg\varphi$.

- <u>for $\exists x\, \varphi(x)$:</u>
  - both ① ⇒ ② and ③ ⇒ ① similar to atomic case. ✓

- Here's our main theorem about $\Vdash^*$

<u>Theorem</u> Sps M is a ctm of ZFC, $\mathbb{P} \in M$, $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$, and G is $\mathbb{P}$-generic$/M$.

<u>Then</u>:

1. if $p \in G$ and $(p \Vdash^* \varphi(\tau_1, \ldots, \tau_n))^M$ then $\varphi(\tau_{1G}, \ldots, \tau_{nG})^{M[G]}$
2. if $\varphi(\tau_{1G}, \ldots, \tau_{nG})^{M[G]}$ then $\exists p \in G$ $(p \Vdash^* \varphi(\tau_1, \ldots, \tau_n))^M$.