- Here's our main theorem about $\Vdash^*$

<u>Theorem</u> Sps M is a ctm of ZFC, $\mathbb{P} \in M$, $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$, and G is $\mathbb{P}$-generic$/M$.

<u>Then</u>:

① if $p \in G$ and $(p \Vdash^* \varphi(\tau_1, \ldots, \tau_n))^M$ then $\varphi(\tau_{1G}, \ldots, \tau_{nG})^{M[G]}$

② if $\varphi(\tau_{1G}, \ldots, \tau_{nG})^{M[G]}$ then $\exists p \in G$ $(p \Vdash^* \varphi(\tau_1, \ldots, \tau_n))^M$.

<u>Pf</u>: - we induct on construction of φ, proving ①+② for a given φ at each stage.

- at a given stage we induct on name rank.

<u>When φ is $\tau_1 = \tau_2$</u>

① sps $p \in G$ and $p \Vdash^* \tau_1 = \tau_2$

- we prove $\tau_{1G} \subseteq \tau_{2G}$ using a.i of def'n of $\Vdash^*$
- $\tau_{1G} \supseteq \tau_{2G}$ is symmetric using a.ii, giving $=$

- every el't of τ_{1G} is π_{1G} for some $\langle \pi_1, s_1 \rangle \in \tau_1$, $s_1 \in G$.
- w.t.s. $\pi_{1G} \in \tau_{2G}$.
- fix $r \in G$, $r \leq p$, $r \leq s_1$ (G a filter)
- by prev. lemma $r \Vdash^* \tau_1 = \tau_2$
- since G generic$/M$ and $G \ni r$, it intersects the dense set below r from

def'n a.i of $r \Vdash^* \tau_1 = \tau_2$

- i.e. can find $q \in G$, $q \leq r$ s.t.
  $\exists \langle \pi_2, s_2 \rangle \in \tau_2 \ (q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)$
- then $s_2 \in G$ too, so $\pi_{2G} \in \tau_{2G}$
- since $q \Vdash^* \pi_1 = \pi_2$, and ranks of $\pi_1, \pi_2 <$ max rank of τ_1, τ_2, by recursion we have $\pi_{1G} = \pi_{2G}$
- hence $\pi_{1G} \in \tau_{2G}$, as desired. ✓

② Now sps $\tau_{1G} = \tau_{2G}$ (ie $e^{M[G]}$ holds)

- w.t.s there is $p \in G$ s.t. $p \Vdash^* \tau_1 = \tau_2$

- let D denote the set of all $r \in \mathbb{P}$ s.t. either
  (x.) $r \Vdash^* \tau_1 = \tau_2$, or
  (y.) $\exists \langle \pi_1, s_1 \rangle \in \tau_1 \ (r \leq s_1 \wedge \forall \langle \pi_2, s_2 \rangle \in \tau_2$
  $\ldots \forall q \in \mathbb{P} [(q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2) \Rightarrow q \perp r])$

  there is π_1 that r puts in τ_1 s.t.
  $\forall q$ compatible w/ r that puts $\pi_2 \in \tau_2$
  does <u>not</u> $\Vdash^* \pi_1 = \pi_2$
  (z.) $\exists \langle \pi_2, s_2 \rangle \in \tau_2 \ (r \leq s_2 \wedge \forall \langle \pi_1, s_1 \rangle \in \tau_1$
  $\ldots \forall q \in \mathbb{P} [(q \leq s_1 \wedge q \Vdash^* \pi_2 = \pi_1) \Rightarrow q \perp r])$

- first: we claim no $r \in G$ can satisfy (y.) or (z.)
- sps $r \in G$ and $\exists \langle \pi_1, s_1 \rangle \in \tau_1$ as in (y.)
- then $s_1 \in G$ so $\pi_{1G} \in \tau_{1G} = \tau_{2G}$
- fix $\langle \pi_2, s_2 \rangle \in \tau_2$ w/ $s_2 \in G$ and $\pi_{1G} = \pi_{2G}$

- then by ② for $\pi_1 = \pi_2$ (recursion on rank)
  $\exists q_0 \in G$ s.t. $q_0 \Vdash^* p_1 = p_2$
- now find $q \leq q_0, s_2$ w/ $q \in G$
- then $q \Vdash^* \pi_1 = \pi_2$ too (prev. lemma)
- hence $q \perp r$ by (y.)
- but $q, r \in G$, contradiction
- (z.) symmetric ✓

- notice $D \in M$ (def'n is abs.)
- thus: if we show D is dense, then $D \cap G \neq \emptyset$ and any $r \in D \cap G$ has $r \Vdash^* \tau_1 = \tau_2$

- We show D is dense
- fix $p \in \mathbb{P}$
- if $p \Vdash^* \tau_1 = \tau_2$ then $p \in D$ and done
- if $p \not\Vdash^* \tau_1 = \tau_2$, then one of a.i and a.ii from def'n of $\Vdash^*$ fails for p.
- sps it is a.i
- then for some $\langle \pi_1, s_1 \rangle \in \tau_1$, the set in def'n of a.i is <u>not</u> dense below p.
- ie $\exists r \leq p$ s.t. $\forall q \leq r$
  $(q \leq s_1 \wedge \forall \langle \pi_2, s_2 \rangle \in \tau_2 (\neg(q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2))$

- in particular, $r \leq s_1$
- given $\langle \pi_2, s_2 \rangle \in \tau_2$ and $q \leq s_2$ s.t. $q \Vdash^* \pi_1 = \pi_2$, must have $q \perp r$:

since any common extension $q' \leq q, r$ would contradict the above.

- thus $r \leq p$ and satisfies (y.) hence is in b
- symmetrically if a.ii fails can find $r \leq p$ satisfying (z.)

$\Rightarrow P$ is dense, as claimed ✓

<u>$\mathcal{C}$ is $\tau_1 \in \tau_2$:</u>

① sps $p \in G$ and $p \Vdash^* \tau_1 \in \tau_2$

- then by def'n:

$$D = \{q : \exists \langle \pi, s \rangle \in \tau_2 \, (q \leq s \wedge q \Vdash^* \pi = \tau_1)\}$$

is dense below p

- so can find $q \in G \cap D$ w/ corresp. $\langle \pi, s \rangle \in \tau_2$ s.t. $q \leq s \wedge q \Vdash^* \pi = \tau_1$
- since $q \in G$, $s \in G$ too $\Rightarrow \pi_G \in \tau_{2G}$
- since $q \Vdash^* \pi = \tau_1$ we have (by ① applied to "$\pi = \tau_1$") $\pi_{1G} = \tau_{1G}$ $M[G]$
$\Rightarrow \tau_{1G} \in \tau_{2G}$, i.e. $(\tau_1 \in \tau_2)^{M[G]}$ ✓

② sps $\tau_{1G} \in \tau_{2G}$

- then $\exists \langle \pi, s \rangle \in \tau_2$ s.t. $\pi_G = \tau_{1G}$, $s \in G$
- by ② for fmla $\pi = \tau_1$, can find $r \in G$ $r \Vdash^* \pi = \tau_1$
- find $p \leq r, s$ $p \in G$
- then $\forall q \leq p$ (in particular for densely many $q \leq p$) we have $q \leq s$ and $q \Vdash^* \pi = \tau_1$

$\Rightarrow p \Vdash^* \tau_1 \in \tau_2$ by def'n ✓

- now we induct on construction of φ
- will suppress writing names
- now have to relativize to M since def'n of $p \Vdash^* \varphi$ no longer absolute.

<u>$\neg\varphi$</u> - sps ①, ② hold for φ; we prove ①, ② for $\neg\varphi$

① sps $p \in G$ and $(p \Vdash^* \neg\varphi)^M$
- w.t.s. $\neg\varphi$ true in $M[G]$
- by def'n of $(p \Vdash^* \neg\varphi)^M$, there is no $r \leq p$ s.t. $(r \Vdash^* \varphi)^M$
- but if $(\varphi)^{M[G]}$ then by ② applied to φ, $\exists q \in G$ s.t. $(q \Vdash^* \varphi)^M$
- then can find $r \leq q, p$ s.t. $r \in G$
- but then $(r \Vdash^* \varphi)^M$, contr.

② sps $\neg\varphi^{M[G]}$ holds
- let $D = \{p : (p \Vdash^* \varphi \vee p \Vdash^* \neg\varphi)^M\}$
- then $D \in M$
- observe that D is dense: given $p \in \mathbb{P}$, by def'n of $\Vdash^*$ (in M), either $(p \Vdash^* \varphi)^M$ in which case $p \in D$, or $\exists q \leq p$ s.t. $(q \Vdash^* \neg\varphi)^M$, in which case $q \in D$.
- hence $D \cap G \neq \emptyset$, so fix $p \in D \cap G$
- if $(p \Vdash^* \neg\varphi)^M$ then done
- if $(p \Vdash^* \varphi)^M$ then by ② applied to φ, $\varphi^{M[G]}$ holds, contradiction $\lightning$

$\underline{\varphi \wedge \psi}$ sps ①, ② hold for φ, ψ; we prove for $\varphi \wedge \psi$.

① sps $p \in G$ and $(p \Vdash^* \varphi \wedge \psi)^M$
- then by def'n of $\Vdash^*$ in M, $(p \Vdash^* \varphi)^M$ and $(p \Vdash^* \psi)^M$
- so by ⓪ $\varphi^{M[G]}$ $\psi^{M[G]}$ hold
$\Rightarrow (\varphi \wedge \psi)^{M[G]}$ holds.

② sps $(\varphi \wedge \psi)^{M[G]}$
- then $\varphi^{M[G]}$, $\psi^{M[G]}$
- so by ② $\exists p, p' \in G$, $(p \Vdash^* \varphi)^M$
$(p' \Vdash^* \psi)^M$
- find $q \in G$, $q \leq p, p'$
- then $(q \Vdash^* \varphi)^M$, $(q \Vdash^* \psi)^M$
$\Rightarrow q \Vdash^* \varphi \wedge \psi$ ✓

$\underline{\exists x\, \varphi(x)}$: sps ①, ② hold for $\varphi(x)$; we prove for $\exists x\, \varphi(x)$

① sps $p \in G$ and $(p \Vdash^* \exists x\, \varphi(x))^M$
- then by def'n of $\Vdash^*$ in M
$$D = \{r : \exists \sigma \in M^{\mathbb{P}}\ (r \Vdash^* \varphi(\sigma))^M\}$$
is dense below p
- also $p \in M$
- so can find $r \in D \cap G$, $r \leq p$ and corr. $\sigma \in M^{\mathbb{P}}$ s.t. $(r \Vdash^* \varphi(\sigma))^M$

- by ① for $\varphi(\sigma)$ we have $\varphi(\sigma)^{M[G]}$
- hence $(\exists x\, \varphi(x))^{M[G]}$

② sps $(\exists x\, \varphi(x))^{M[G]}$
- then for some σ_G we have $\varphi(\sigma_G)^{M[G]}$
- by ② for $\varphi(\sigma)$, $\exists p \in G, (p \Vdash^* \varphi(\sigma))^M$
- then $\forall q \leq p$ (in particular for densely many $q \leq p$ we have $(q \Vdash^* \varphi(\sigma))^M$
- hence $(p \Vdash^* \exists x\, \varphi(x))^M$ ✓

Finally we can formally state Facts 1 and 2.

<u>Theorem</u> Sps M is a ctm of ZFC, $\mathbb{P} \in M$. Sps $\varphi(x_1, \ldots, x_n)$ a fmla and $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$.

<u>Then</u>: ① $\forall p \in \mathbb{P}$

$p \Vdash \varphi(\tau_1, \ldots, \tau_n)$ iff $(p \Vdash^* \varphi(\tau_1, \ldots, \tau_n))^M$

② $\forall G$ that are $\mathbb{P}$-generic $/M$

$\varphi(\tau_{1G}, \ldots, \tau_{nG})^{M[G]}$ iff $\exists p \in G. (p \Vdash \varphi(\tau_1, \ldots, \tau_n))$

<u>Pf</u> ① the $\Leftarrow$ implication immediate from ① of prev. thm
- <u>for $\Rightarrow$</u>: sps $p \Vdash \varphi(\tau_1, \ldots, \tau_n)$
- to show $(p \Vdash^* \varphi(\tau_1, \ldots, \tau_n))^M$, suff. to show

$$D = \{r : (r \Vdash^* \varphi(\tau_1, \ldots, \tau_n))^M\}$$

dense below p

- if not, can find $q \leq p$ s.t. $\nexists\, r \leq q,\ r \in D$
- then by def'n of $\Vdash^*$

$$(q \Vdash^* \neg\varphi(\tau_1, \ldots, \tau_n))^M$$

- so by $\Leftarrow$ direction we have $q \Vdash \neg\varphi(\tau_1, \ldots, \tau_n)$
- fix $G \ni q$ $\mathbb{P}$-generic/M
- then $\neg\varphi(\tau_{1G}, \ldots, \tau_{nG})\ M[G]$
- but $p \in G$ too (since $p \geq q$)

giving $\varphi(\tau_{1G}, \ldots, \tau_{nG})^{M[G]}$, contradiction

② ($\Leftarrow$) immediate from def'n of $\Vdash$

($\Rightarrow$) immediate from ① + prev. thm ✓