- we showed each D_f is dense in $\mathbb{P}$
i.e. $\forall p \in \mathbb{P}\ \exists q \in D_f \quad q \leq p$
- it cannot be there is a filter $G \subseteq \mathbb{P}$
s.t. $G \cap D_f \neq \emptyset$ for <u>every</u> $f \in {}^{\omega}2$
- if there were such a G, could
define $g = \bigcup G$
- g would be a function from ω to
2 different from every function from
ω to 2 ... (hm!)

... but what if we carry out this
construction inside M?

- i.e. let $\mathbb{P}^M = \{p \in M : p$ is a finite
partial function from ω to $2\}$
- $\mathbb{P}^M$ is M's version of the actual $\mathbb{P}$
- but actually (since $M \models ZFC$) M can
construct every finite partial function
from ω to 2, so $\mathbb{P}^M = \mathbb{P}$

- on other hand, ${}^{\omega}2$ is uncountable, so
$({}^{\omega}2)^M \neq {}^{\omega}2$
- but "being a function from ω to 2"
is absolute, so we do have $({}^{\omega}2)^M = {}^{\omega}2 \cap M$

- now, $\forall f \in ({}^{\omega}2)^M$ can define (within M,
as it were)
$$D_f^M = \{p \in \mathbb{P}^M : p(n) \neq f(n) \text{ for some } n\}$$

(- can check that $D_f^M = D_f$)
- $D_f \in M$ iff $f \in M$, hence only ctbly many D_f's in M
- thus: from <u>outside</u> of M we can find a filter G (will have $G \subseteq \mathbb{P}$ but $G \notin M$) s.t. $G \cap D_f \neq \emptyset$ for every $f \in M$.
- we know $G \notin M$ by our ex. above!
- in fact we can do better: since M is ctbl, $\exists$ only ctbly many dense $D \subseteq \mathbb{P}^M$ s.t. $D \in M$, so we can find a G that intersects <u>every</u> dense set D of $\mathbb{P}^M$ <u>that</u> <u>belongs</u> <u>to M</u>.

- we call G a <u>generic</u> <u>filter</u> for $\mathbb{P}$ over M.
- if we let $g = \bigcup G$, then $g : \omega \to 2$ is different, not from <u>every</u> $f \in {}^\omega 2$, but every $f \in ({}^\omega 2)^M$. Thus $g \notin M$
- if we view g as a characteristic function for some $A \subseteq \omega$, must have $A \notin M$ too (since each of g, A can be constructed from the other)

- the theory of forcing then allows us to build a model $M[G] \models ZFC$ from M and the generic filter G
- how we actually build $M[G]$, and showing $M[G] \models ZFC$, requires work.

- but intuitively $M[G]$ is the smallest transitive model of ZFC extending M that contains G.
- i.e. the sets in $M[G]$ are the sets the sets that can be formed from the sets in M along with G, using the axioms of ZFC.
- then we ask: how much have things changed in passing from M to $M[G]$?

- e.g. we have a new $g: \omega \to 2$, hence we have a new corresponding $A \subseteq \omega$.
- are there other new subsets of ω? (yes, e.g. any finite modification of A will be new and in $M[G]$)

- by absoluteness, every ordinal $\alpha \in M$ is still in $M[G]$ and is still an ordinal.
- are there new ordinals in $M[G]$?

- "being a cardinal" is <u>not</u> absolute, as we've seen.
- in going from M to $M[G]$, have the cardinals changed?
  e.g. do we have $\omega_1^M = \omega_1^{M[G]}$?

- to answer these q's we need to develop theory of forcing.
- once we do, we'll see that the poset $\mathbb{P}$ that we choose in the ground model affects the properties of the generic extension $M[G]$.

- we'll be able to show, e.g., there is a $\mathbb{P}$ s.t. if G is generic for $\mathbb{P}$ over M, then $M[G] \models ZFC + \neg CH$.
- thus $ZFC + \neg CH$ is consistent! ... assuming we can find $M \models ZFC$ to begin with.

So: our rough plan for the class is to expand on these topics:

① Absolutenes and transitive models of set theory

② forcing and generic extensions

- we asserted at beginning of discussion:

$Con(ZFC) \Rightarrow \exists$ a ctbl model $\langle N, E \rangle \models ZFC$

116a + ignoring some metamath issues

$\Rightarrow \exists$ a <u>ctbl</u> <u>transitive</u> <u>model</u> $\langle M, \in \rangle \models ZFC$ equipped w/ actual set membership $\in$.

how??

- How do we get $\langle M, \in \rangle$ from $\langle N, E \rangle$?
- idea is:

  replace N's version of "$\emptyset$" with actual $\emptyset$
  N's version of "1" w/ actual $1 = \{\emptyset\}$
  etc...

- for this construction to turn all "sets" in N into actual sets, we need the "sets" in N to be formed by starting with (N's version of) the empty set and inductively applying set theoretic operations.
- but all we know is that $N \models ZFC$
- so really we want ZFC to prove "all sets are formed starting w/ $\emptyset$ and inductively applying set theoretic operations"

- this involves introducing:
  (i) a new idea called well-foundedness
  (ii) a new axiom called the axiom of Foundation

  asserts: "all sets are formed starting w/ $\emptyset$ and taking powersets ω many times."

- then we'll show how we can turn a formal model $\langle N, E \rangle$ of ZFC into a transitive one $\langle M, \in \rangle$ by formalizing idea above.
- then prove the results about what is absolute between M and V (e.g. $\emptyset$, ω, "being an ordinal", etc.) and what is not (e.g. "being ω," "being $P(\omega)$" etc.)

↳ will be a little intuitive in places so we can see big picture w/o getting lost in details
↳ then we'll be ready for forcing!

<u>The well-founded sets</u>

- we introduce the class of well-founded sets (WF)
- WF includes all sets in the hierarchy that is formed by starting w/ $\emptyset$ and transfinitely iterating powerset operation
- WF excludes "paradoxical" sets like $x = \{x\}$

<u>Def'n</u> We define the set $R(\alpha)$ for $\alpha \in ON$ by transfinite recursion.

1) $R(0) = 0 = \emptyset$
2) $R(\alpha+1) = P(R(\alpha))$
3) $R(\alpha) = \bigcup_{\beta<\alpha} R(\beta)$ for α limit

So e.g. $R(1) = P(\emptyset) = \{\emptyset\}$

$R(2) = P(P(\emptyset)) = \{\emptyset, \{\emptyset\}\}$

$R(3) = P(P(P(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

$\vdots$

etc.

$R(\omega) = \bigcup_{n\in\omega} R(n)$

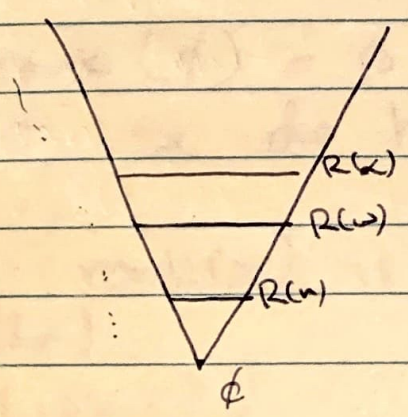
$\vdots$

etc.

<u>Def'n</u> $WF = \bigcup\{R(\alpha) : \alpha \in ON\}$

- Say x is <u>well-founded</u> if $x \in WF$ (i.e. $x \in R(\alpha)$ for some $\alpha \in ON$)

<u>Pic of WF</u>:



<u>Lemma</u> $\forall \alpha \in ON$ we have

1) $R(\alpha)$ is transitive $\leftarrow$ $x \in R(\alpha) \Rightarrow x \subseteq R(\alpha)$

2) $\beta \leq \alpha \Rightarrow R(\beta) \subseteq R(\alpha)$

<u>Pf</u>: Transfinite induction (you try!)

↳ it follows WF is a transitive class

i.e. $x \in WF \Rightarrow x \subseteq WF$ (since $x \in R(\alpha)$ for some α hence $x \subseteq R(\alpha)$)

↳ in fact the $R(\alpha)$'s are strictly increasing, i.e. $\beta < \alpha \Rightarrow R(\beta) \subsetneq R(\alpha)$ (why?)

- even better: we can rank the sets in WF according to first place in the hierarchy they appear.
- notice that (by def'n), for $x \in WF$, the least α s.t. $x \in R(\alpha)$ is a <u>successor</u>.

<u>Def'n</u> If $x \in WF$, $\text{rank}(x) :=$ least $\beta \in ON$ s.t. $x \in R(\beta+1)$.

e.g. $\text{rank}(\emptyset) = 0$ since, while $0 \notin R(0) = 0 = \emptyset$, we do have $0 \in R(1) = \{\emptyset\}$

<u>Observe</u>: $\text{rank}(x)$ is the least α s.t. $x \subseteq R(\alpha)$

(why?)