<u>Lemma</u> $\forall \alpha$ we have $R(\alpha) = \{x \in WF : \text{rank}(x) < \alpha\}$

Pf: Think about it!

<u>Lemma</u> Given $y \in WF$ we have
1) $x \in y \Rightarrow \text{rank}(x) < \text{rank}(y)$
2) $\text{rank}(y) = \sup\{\text{rank}(x) + 1 : x \in y\}$

Pf: you try!
(awkwardness of the $\text{rank}(x)+1$ in the sup is because if there is $x \in y$ of maximum rank then $\text{rank}(y) = \text{rank}(x)+1$ and if there is no such x then $\text{rank}(y) = \sup\{\text{rank}(x) : x \in y\} = \sup\{\text{rank}(x)+1 : x \in y\}$)

- Intuitively, because sets in WF are constructed out of sets of lower rank there are no "paradoxical" sets in WF.
- e.g if $x \in WF$, can't have $x \in x$ since by ① of lemma this would give $\text{rank}(x) < \text{rank}(x)$

- all ordinals are in WF and are their own rank:

<u>Lemma</u> $\forall \alpha \in ON$ we have:

1. $\alpha \in WF$ and $\text{rank}(\alpha) = \alpha$
2. $R(\alpha) \cap ON = \alpha$

Pf: 1 = transfinite induction (you try!)
2 = immediate from 1.

- WF also includes all of the sets we've built from the ordinals, since WF is closed under all familiar set theoretic operations
(- we'll argue later that in fact $WF \models ZFC$)

<u>Lemma</u> a) if $x \in WF$, then so is $\bigcup x$, $P(x)$, $\{x\}$ and the ranks of these sets are all $< \text{rank}(x) + \omega$
b) if $x, y \in WF$, then so are $x \times y$, $x \cup y$, $x \cap y$, $\{x, y\}$, $\langle x, y \rangle$, and ${}^y x$, and the ranks of these sets are all $< \max(\text{rank}(x), \text{rank}(y)) + \omega$.

Pf: - we'll do a) and leave b) as an exercise
- fix $x \in WF$
- sps $\text{rank}(x) = \alpha$, so that $x \subseteq R(\alpha)$
- hence $P(x) \subseteq P(R(\alpha)) = R(\alpha+1)$
and $P(P(x)) \subseteq P(R(\alpha+1)) = R(\alpha+2)$

- since $\{x\}$ and $P(x) \in P(P(x))$ we have $\{x\}, P(x) \in R(\alpha+2)$

- For $\bigcup x = \bigcup_{y \in x} y$, notice that since $x \subseteq R(\alpha)$ we have $y \in R(\alpha)$ $\forall y \in x$
- hence $y \subseteq R(\alpha)$ $\forall y \in x$, by transitivity
- hence $\bigcup x = \bigcup_{y \in x} y \subseteq R(\alpha)$

$\Rightarrow \bigcup x \in P(R(\alpha)) = R(\alpha+1)$ ✓

<u>Lemma</u> $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all in WF (in fact in $R(\omega+\omega)$)

Pf: we constructed these sets from ω (which is $\in R(\omega+1)$) using finitely many operations from the prev. lemma (and taking subsets). ✓

- $R(\omega) = \bigcup_{n \in \omega} R(n)$ is sometimes called the class of <u>hereditarily</u> <u>finite</u> <u>sets</u>
- a few facts:

<u>Lemma</u> ① $R(n)$ is finite $\forall n \in \omega$
② $|R(\omega)| = \omega$

<u>Pf</u>: ① by induction on n (what is $|R(\omega)|$?)

② follows from ① ✓

- once we get past ω, the cardinalities of the $R(\alpha)$'s grow exponentially
- e.g. $|R(\omega+1)| = |P(R(\omega))| = 2^{\omega}$
  $|R(\omega+2)| = |P(R(\omega+1))| = 2^{2^{\omega}}$
  $\vdots$ etc.

- one can show that all of the mathematical objects we're familiar with appear in WF (or at least have copies in WF)
- e.g. every group, ring, topological space etc. is in WF
- again, we'll "prove" later that WF $\models$ ZFC.

<u>Well-founded relations</u>

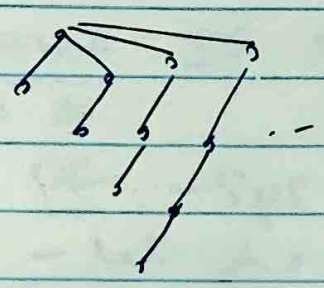
- well-founded relations generalize well-orders.

<u>Def'n</u> A relation R is <u>well-founded</u> on a set A if every nonempty subset $X \subseteq A$ has a (i.e. at least one) R-least element.

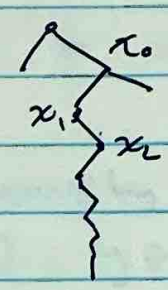
i.e. $\forall X \subseteq A \, (X \neq \emptyset \Rightarrow \exists x \in X \, \forall y \in X \, (y \not R x))$

<u>Ex's</u> ① if R is a well-order on A then R is well-founded on A

② Consider the following trees, where xRy if x is a node below y



a well-founded tree

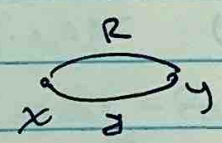


inf descending seq.

$\ldots x_2 R x_1 R x_0$

a non-well-founded tree

③ loops are not well-founded, e.g. if $\exists x \neq y \in A$ s.t. $xRyRx$ then $\{x,y\}$ has no least el't.



④ In general: R is not well-founded on A iff we can find either an R-loop or an infinite descending R-chain in A (why?)

- are the sets in WF well-founded in some way? Yes! w.r.t. the $\in$-relation
- can see this since we have a rank for sets in WF

<u>Lemma</u> If $A \in WF$ then $\in$ is well-founded on A.

<u>Pf</u>: - Sps $X \subseteq A$ nonempty
- let $\alpha = \min\{\text{rank}(y) : y \in X\}$ ← ranks are ordinals!
- fix $y \in X$ s.t. $\text{rank}(y) = \alpha$
- if $x \in y$ then $\text{rank}(x) < \text{rank}(y)$ by above hence $x \notin X$ by minimality of α.

$\Rightarrow$ y is $\in$-minimal in X ✓

- Converse to lemma not necessarily true.
- e.g. sps we had sets $x = \{y\}$ and $y = \{x\}$
- then $x, y \notin WF$ (can't be ranked) but $\in$ is well-founded on x (and also y) ← in fact empty.

- A partial converse:

<u>Lemma</u> if A is transitive and $\in$ is well-founded on A, then $A \in WF$.

To prove lemma, let's first verify:

<u>Fact</u>: for every set x:

$$x \in WF \iff x \subseteq WF$$

<u>Pf</u>: $\Rightarrow$ is clear, why so sps $x \subseteq WF$

- let $\alpha = \sup\{\text{rank}(y)+1 : y \in x\}$
- then $\text{rank}(y) < \alpha \quad \forall y \in x$
- hence $x \subseteq R(\alpha)$
- hence $x \in R(\alpha+1)$

<u>Pf of lemma</u>: - sps toward contradiction A is transitive and $\in$ is well-founded on A but $A \notin WF$

- then $A \notin WF$ by fact, so let $X = \{x \in A : x \notin WF\}$
- X is nonempty, so fix $y \in X$ that is $\in$-minimal
- every $x \in y$ is $\in A$ by transitivity but $x \notin X$ by minimality of y
- hence $y \subseteq WF$
  $\Rightarrow y \in WF$ by fact. Contradiction ✓

↳ The idea is not that WF is the largest class on which $\in$ is well-founded but the largest <u>transitive</u> class on which $\in$ is well-<u>founded</u>.

- i.e. we can't find $A_0 \in WF$ and sets $A_1, A_2, \ldots$ s.t.

$A_0 \ni A_1 \ni A_2 \ni \cdots$

or $A_0 \ni A_1 \ni A_2 \ni \cdots \ni A_n \ni A_0$

- this is made precise by concept of transitive closure.

<u>Def'n</u> Fix a set A

① By recursion on $n \in \omega$, define:

$$\bigcup^0 A = A$$
$$\bigcup^{n+1} A = \bigcup(\bigcup^n A)$$

② Define: the <u>transitive closure</u> of A is

$$\mathrm{trcl}(A) := \bigcup\{\bigcup^n A : n \in \omega\}$$
$$= A \cup \bigcup A \cup \bigcup\bigcup A \cup \cdots$$

- Concretely: $\mathrm{trcl}(A)$ contains exactly:

the el'ts of A
the el'ts of el'ts of A
the el'ts of el'ts of el'ts of A
etc.

e.g. if $A = \{\{\{\emptyset\}\}\}$
then $\mathrm{trcl}(A) = \{\{\{\emptyset\}\}, \{\emptyset\}, \emptyset\}$

<u>Lemma</u> ⓐ $A \subseteq \mathrm{trcl}(A)$
ⓑ $\mathrm{trcl}(A)$ is transitive
ⓒ if $A \subseteq T$ and T is transitive then $\mathrm{trcl}(A) \subseteq T$

(i.e. $trcl(A)$ is smallest transitive set containing A as a subset)

d) if A is transitive, $trcl(A) = A$
e) if $x \in A$ then $trcl(x) \subseteq trcl(A)$
f) $trcl(A) = A \cup \{trcl(x) : x \in A\}$

Pf: you try!

- the following explains why we called $\bigcup_{\alpha \in ON} R(\alpha)$ the class of well-founded sets (WF)

<u>Theorem</u> For any set A, TFAE:
a) $A \in WF$
b) $trcl(A) \in WF$
c) $\in$ is well-founded on $trcl(A)$

<u>Pf:</u> <u>a) $\Rightarrow$ b)</u>: if $A \in WF$ then since WF is transitive we have $\bigcup^n A \in WF$ $\forall n \in \omega$ (hence $\subseteq$ and $\subseteq$)
hence $trcl(A) \subseteq WF$ hence $trcl(A) \in WF$

<u>b) $\Rightarrow$ c)</u> if $trcl(A) \in WF$ then $\in$ is well founded on $trcl(A)$ by above

<u>c) $\Rightarrow$ a)</u>: since $trcl(A)$ is transitive we have $trcl(A) \in WF$ iff $\in$ is

well-founded on trcl(A), by prev. lemmas
- hence trcl(A) $\in$ WF
$\Rightarrow$ trcl(A) $\subseteq$ WF $\Rightarrow$ A $\subseteq$ WF $\Rightarrow$ A $\in$ WF ✓

The Axiom of Foundation

- we now come to our final axiom of ZFC.
- it asserts "V = WF" i.e. the universe of sets consists exactly of the well-founded sets

- this is not an axiom that affects mathematical practice:
  ↳ we've seen that all math'l objects (or copies of them) belong to WF
  ↳ however, existence of non-well-founded sets like $x = \{x\}$ is consistent w/ other axioms
  ↳ allowing such sets wouldn't affect math, but would be less convenient for us.

- actually writing "V = WF" in lang. of set theory would be a pain
- fortunately (as we've more or less shown) there is a simple equivalent statement. namely "every nonempty set has a $\in$-least element"