well-founded on trcl(A), by prev. lemmas
- hence trcl(A) $\in$ WF
$\Rightarrow$ trcl(A) $\subseteq$ WF $\Rightarrow$ A $\subseteq$ WF $\Rightarrow$ A $\in$ WF ✓

The Axiom of Foundation

- we now come to our final axiom of ZFC.
- it asserts "V = WF" i.e. the universe of sets consists exactly of the well-founded sets

- this is not an axiom that affects mathematical practice:
  ↳ we've seen that all math'l objects (or copies of them) belong to WF
  ↳ however, existence of non-well-founded sets like $x = \{x\}$ is consistent w/ other axioms
  ↳ allowing such sets wouldn't affect math, but would be less convenient for us.

- actually writing "V = WF" in lang. of set theory would be a pain
- fortunately (as we've more or less shown) there is a simple equivalent statement. namely "every nonempty set has a $\in$-least element"

<u>Axiom 2</u> (Foundation)

$$\forall x\ (x \neq \emptyset \Rightarrow \exists y \in x\ (\neg \exists z \in x\ (z \in y)))$$

- axiom is equiv to:

  if $x \neq \emptyset$ then $\exists y \in x$ s.t. $x \cap y = \emptyset$.

<u>Theorem</u> TFAE:

a) Axiom of Foundation

b) for every set A, $\in$ is well-founded on A

c) $V = WF$

<u>Pf</u>: <u>(a) $\Leftrightarrow$ (b)</u>: immediate from def'n of well founded applied to $\in$

<u>b) $\Rightarrow$ c)</u>: b) implies $\in$ is w.f. on $trcl(A)$ for every A, which $\Rightarrow A \in WF$

<u>c) $\Rightarrow$ a)</u>: by prev. lemma.

- though Foundation does not change any of the math we've developed on the basis of the other axioms (all in WF anyway), it does make certain def'ns easier, e.g.

<u>Prop'n</u>: x is an ordinal iff x is transitive and totally ordered by $\in$

<u>Pf</u>: if $\in$ a total order on x and $x = trcl(x)$ is transitive, then by Foundation every $y \subseteq x$

has a $\in$-least el't, i.e. x is <u>well</u>-ordered by $\in$ ✓

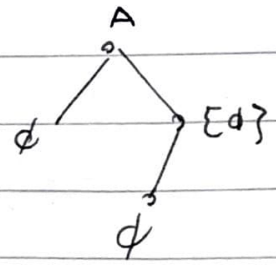
<u>Induction and recursion on well-founded relations</u>

- we can generalize induction/recursion on well-orders to well-founded relations
- understanding w.f. induction/recursion can be aided by <u>tree diagrams</u>

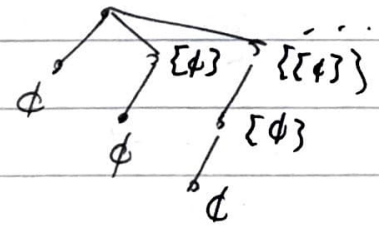
- Sps we have a set A
- we can draw a tree diagram as follows
  - root node represents A
  - descendants of root represent el'ts of A
  - descendants of these represent el'ts of el'ts of A

etc..

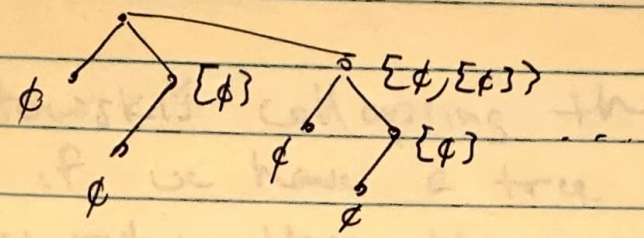
e.g. ① if $A = \{\emptyset, \{\emptyset\}\}$



② if $A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\}$



③ $A = \omega = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots\}$



- the diagram is a way of visualizing trcl(A)
  - first level of tree = el'ts of A
  - second level = el'ts of el'ts of A
    etc.
- different nodes may represent the same set: e.g. every terminal node is $\emptyset$.
- if A = trcl(A) is transitive, <u>all</u> el'ts of trcl(A) appear in first level.

- can't have two nodes labelled w/ same set be related (no $\in$-loops)
- can't have infinite branches (no $\in$-descending chains)
- can't have two nodes w/ same label share a parent node (extensionality)

↳ if we erase the labels of the tree, notice the diagram is governed by following rules:

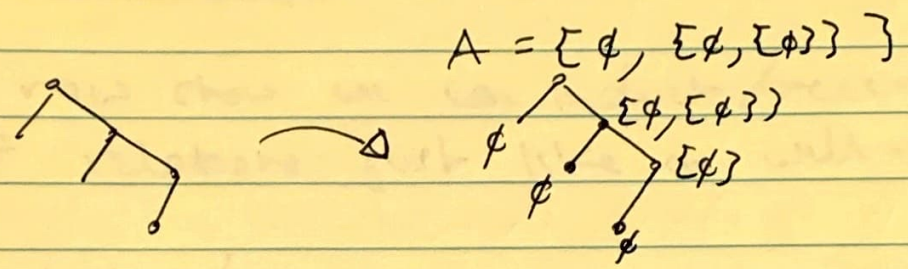
① no infinite branches

② if y, z direct descendants of a parent node x, then trees below

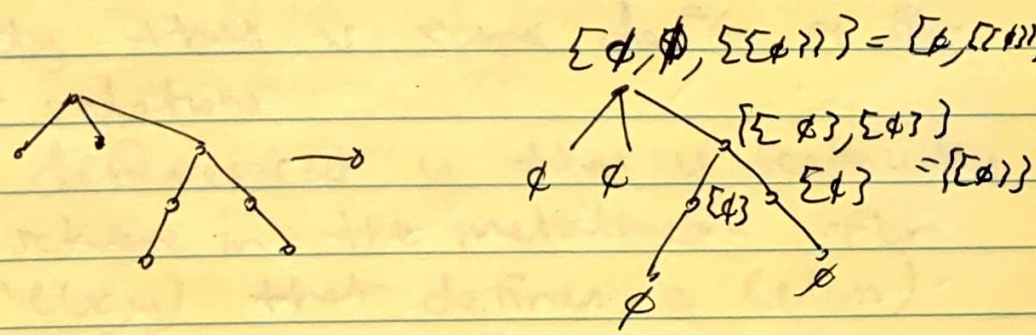
y, z can't be isomorphic (this is extensionality)

- Mostowski's collapsing theorem is essentially that if we have a tree that satisfies these rules then there is a unique set which has the tree as its $\in$-diagram.

- e.g.



- and even if we don't satisfy second rule, a well-founded tree (i.e. no inf. branches) still leads to a set, but now we might have to "collapse" nodes that represent the same set as we go up, eg.:



- We can even draw such a diagram for a <u>class</u> $\underset{\sim}{A}$ of sets

- only difference w first level will constitute a class of sets (namely $\underset{\sim}{A}$)

<u>Induction / Recursion on well-founded (class) relations</u>

- we'll now show we can induct/recurse on w.f. relations just like on well-orders

- w.f. induction/recursion makes sense for classes as well as sets.

<u>Def'n</u> A (class) relation $\underset{\sim}{R}$ is <u>well-founded</u> on a class $\underset{\sim}{A}$ if every nonempty <u>set</u> $X \subseteq \underset{\sim}{A}$ has an R-minimal el't.

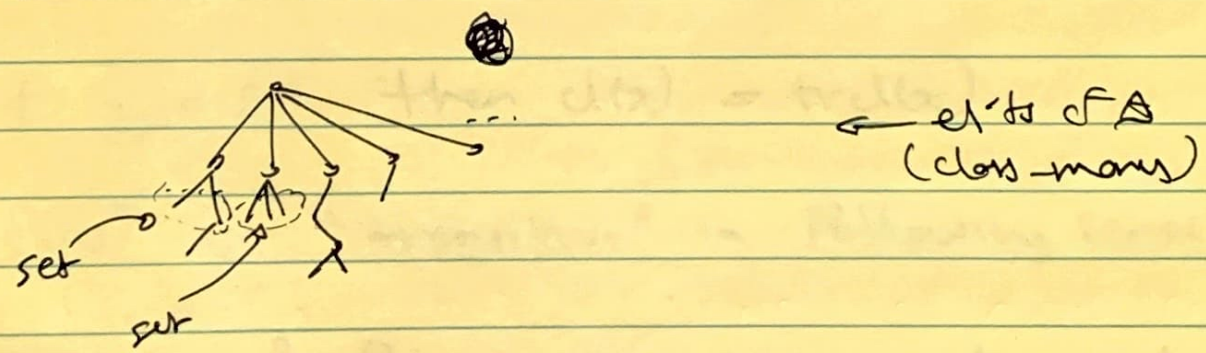
- informally this is same def'n as for w.f. set relations
- formal difference is this is technically a def'n scheme in the metatheory: for every $\varphi(x,y)$ that defines a (class) relation $\underset{\sim}{R}$, and every $\psi(z)$ that defines a class $\underset{\sim}{A}$, say $\underset{\sim}{R}$ is w.f. on $\underset{\sim}{A}$ if for every set $X \subseteq \underset{\sim}{A}$ $\exists x \in X$ s.t. $\forall y \in X$ we have $\neg \varphi(x,y)$

- to make proofs by induction/recursion make sense we usually insist on an extra condition:

<u>Def'n</u> $\underset{\sim}{R}$ is <u>set-like</u> on $\underset{\sim}{A}$ if for all $x \in A$, $\{y \in A : yRx\}$ is a set.

- just like for $\in$, we can draw tree diagrams for a w.f. class relation $\underset{\sim}{R}$ on $\underset{\sim}{A}$
- since $\underset{\sim}{R}$ w.f., these diagrams will have no infinite branches
- if $\underset{\sim}{R}$ is set-like on $\underset{\sim}{A}$, first level of diagram may be a (proper) class namely $\underset{\sim}{A}$, but collection of descendants for any other node is a set

i.e. collection of descendants of root



- e.g. can think of $\in$ as a w.f. class relation on V — will be set-like!

- if we think of a given w.f. $\underset{\sim}{R}$ as akin to $\in$, then $\{y \in \underset{\sim}{A} : yRx\}$ is akin to the set of el'ts in x (i.e. x)

- more formally we define:

<u>Def'n</u>: Sps $\underset{\sim}{R}$ is set-like on $\underset{\sim}{A}$.

① $\text{pred}(x) := \{y \in \underset{\sim}{A} : y R x\}$

② $\text{pred}^0(x) = \text{pred}(x)$

$\text{pred}^{n+1}(x) = \bigcup \{\text{pred}(y) : y \in \text{pred}^n(x)\}$

③ $\text{cl}(x) = \bigcup \{\text{pred}^n(x) : n \in \omega\}$

<u>note</u>: should really write something like $\text{pred}(x, \underset{\sim}{R}, \underset{\sim}{A})$ but...

- ③ in words: $\text{cl}(x)$ consists of
  - $\underset{\sim}{R}$-predecessors of x
  - $\underset{\sim}{R}$-preds of R-preds of x
  - ... etc.

- if $\underset{\sim}{R} = \in$ then $\text{cl}(x) = \text{trcl}(x)$

- $\text{cl}(x)$ is "transitive" in following sense:

<u>Lemma</u> if $\underset{\sim}{R}$ is set-like on $\underset{\sim}{A}$ and $x \in \underset{\sim}{A}$ then $\forall y \in \text{cl}(x)$ we have $\text{pred}(y) \subseteq \text{cl}(x)$.

<u>Pf</u>: follows from def'n.

<u>Theorem</u> (Transfinite induction on w.f. relations)
If $\underset{\sim}{R}$ is w.f. and set-like on $\underset{\sim}{A}$ then every nonempty subclass $\underset{\sim}{X} \subseteq \underset{\sim}{A}$ has an $\underset{\sim}{R}$-minimal el't.

<u>Pf</u>: - only wrinkle here is translating from classes to sets
- fix $x \in \underset{\sim}{X}$; if x is $\underset{\sim}{R}$-min'l in $\underset{\sim}{X}$, done.
- o/w consider $\underbrace{\underset{\sim}{X} \cap \underbrace{cl(x)}_{\text{a set by set-likeness}}}_{\text{a set by comprehension}}$
- non-empty by assumption and a set
hence $\exists y \in \underset{\sim}{X} \cap cl(x)$ which is $\underset{\sim}{R}$-min'l
- by above y is $\underset{\sim}{R}$-min'l in $\underset{\sim}{X}$ ✓

<u>Theorem</u> (Transfinite recursion on w.f. relations)
Suppose $\underset{\sim}{R}$ is w.f. and set-like on $\underset{\sim}{A}$. If $\underset{\sim}{F}: \underset{\sim}{A} \times \underset{\sim}{V} \to \underset{\sim}{V}$ is a class function then there is a unique $\underset{\sim}{G}: \underset{\sim}{A} \to \underset{\sim}{V}$ s.t. $\forall x \in \underset{\sim}{A}$:

$$\underset{\sim}{G}(x) = \underset{\sim}{F}(x, G \restriction pred(x))$$

Pf: we'll omit, but similar to proof of recursion on ON.

Theorem says: if we have a background function F (that takes as input "el'ts of $\underset{\sim}{A}$" + "set" and outputs "set"), then we can apply F iteratively to $\underset{\sim}{A}$ starting at the $\underset{\sim}{R}$-minimal elements and going up s.t. at a given node x the output is $F(x, \text{"all previous info"})$

i.e. our computation rule is allowed to use x + all info about outputs at predecessors of x to compute $G(x)$

<u>Ex</u> We proved (or really, left you to prove) the identity

$$\text{rank}(y) = \sup\{\text{rank}(x)+1 : x \in y\}$$

for $y \in WF$

- Can also view this as a <u>definition</u> by recursion on the w.f. relation $\in$
- i.e. we might <u>define</u> rank(y) by recursion:

$$\text{rank}(\emptyset) = 0$$
$$\text{rank}(y) = \sup\{\text{rank}(x)+1 : x \in y\}$$

base case (defn for $\in$-min el'ts)

how to compute rank(y) in terms of ranks of $\in$-predecessors