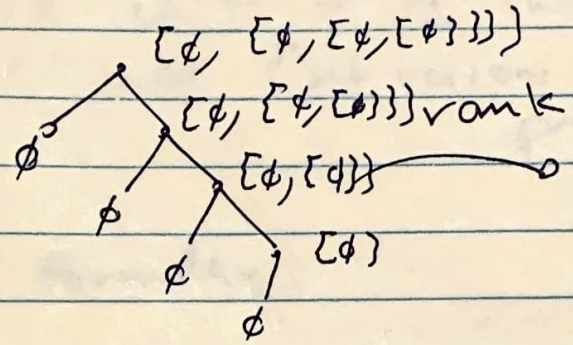
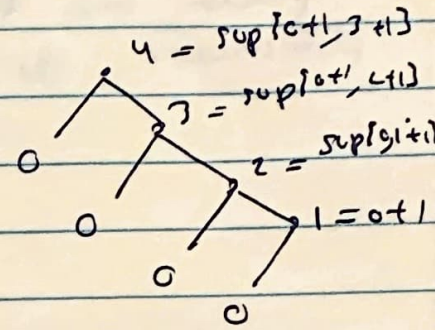
- intuitively this is a well-defined notion of rank for all y that can be built up from $\emptyset$ (i.e. all $y \in WF$)
- Theorem says this can be formalized.



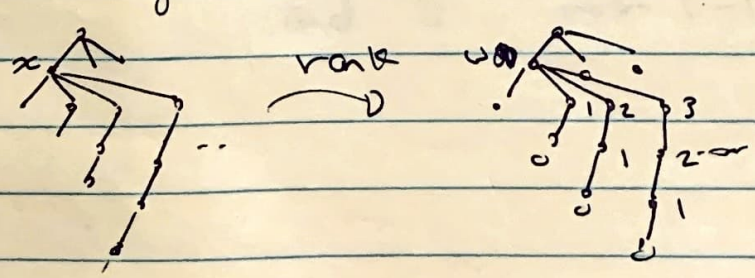


- can generalize notion of rank to an arbitrary w.f. set-like relation

<u>Def'n</u> Sps $\underset{\sim}{R}$ is w.f. and set like on $\underset{\sim}{A}$. Define

$$\text{rank}(x, \underset{\sim}{A}, \underset{\sim}{R}) = \text{rank}_R(x) := \sup\{\text{rank}_R(y)+1 : y \underset{\sim}{R} x \wedge y \in \underset{\sim}{A}\}$$

- this is a def'n by recursion
- e.g. if $\underset{\sim}{R}$ on $\underset{\sim}{A}$ is represented by the diagram



then node x has rank ω $= \sup\{n+1 : n \in \omega\}$

- now we get to an important instance of w.f. recursion: the <u>Mostowski collapse</u>
- idea is simple: given $\underset{\sim}{R}$ on $\underset{\sim}{A}$ can "turn $\underset{\sim}{R}$ into $\in$" by turning every $\underset{\sim}{R}$-min'l el't of $\underset{\sim}{A}$ into $\emptyset$ and every node x into {set versions of its recursively preds}

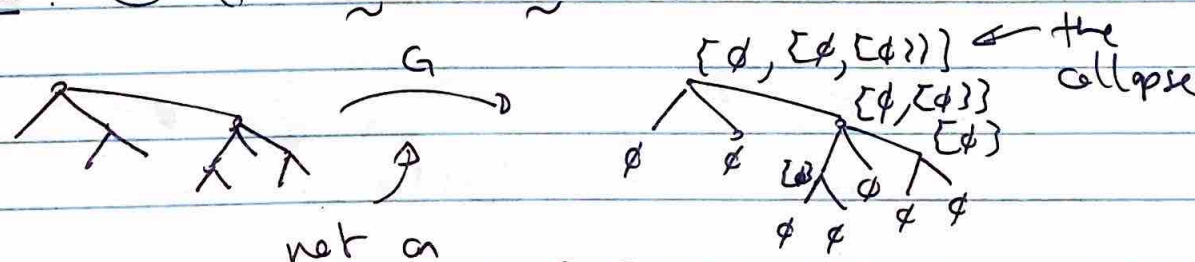
- More formally:

<u>Def'n</u> Sps $\underset{\sim}{R}$ is w.f. and set-like on $\underset{\sim}{A}$. Define G by:

$$G(x) = \{G(y) : y \in \underset{\sim}{A} \wedge y \underset{\sim}{R} x\}$$

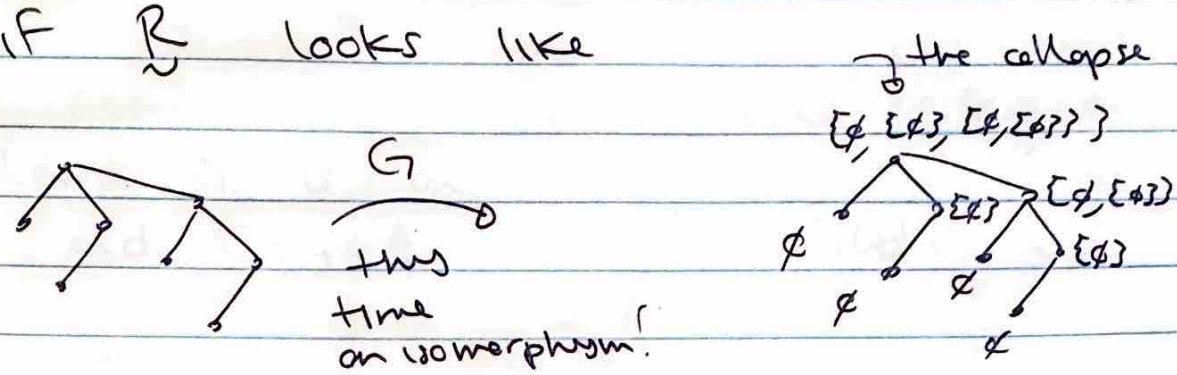
The <u>Mostowski collapse</u> of $\underset{\sim}{A}$ is defined to be $\mathrm{ran}(G)$

<u>Ex</u>: ① If $\underset{\sim}{R}$ on $\underset{\sim}{A}$ looks like:



not an isomorphism of $\underset{\sim}{R}$ and $\in$ .. not 1-1!

② if $\underset{\sim}{R}$ looks like



<u>Lemma</u> ① $\forall x, y \in \underset{\sim}{A}$ we have: $x R y \Rightarrow G(x) \in G(y)$

② M is transitive

③ $M \subseteq WF$

④ $\forall x \in \underset{\sim}{A}$, $\text{rank}_R(x) = \text{rank}(G(x))$

<u>Pf</u>: you try!

- while G may not be 1-1 on $\underset{\sim}{A}$ in general, ④ of Lemma says G <u>does</u> preserve rank

- Here is the condition that guarantees the collapsing function is 1-1.

<u>Def'n</u> $\underset{\sim}{R}$ is <u>extensional</u> on $\underset{\sim}{A}$ if
$\forall x, y \in \underset{\sim}{A} \; (\forall z \in \underset{\sim}{A} \; (z R x \Leftrightarrow z R y) \Rightarrow x = y)$

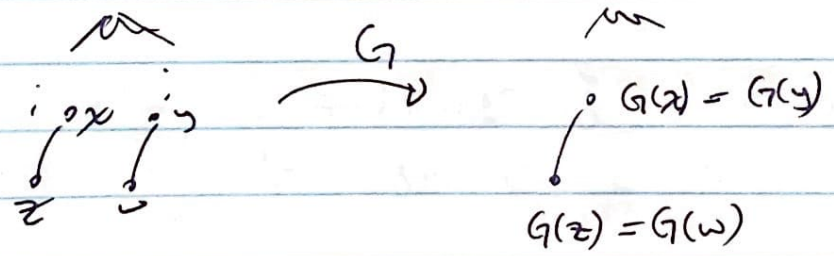
i.e. "the axiom of extensionality holds for $\underset{\sim}{R}$ on $\underset{\sim}{A}$"

<u>Lemma</u>: If $\underset{\sim}{R}$ is extensional, well-founded, and set-like on $\underset{\sim}{A}$ then the Mostowski collapse G is an isomorphism, i.e. G is 1-1 and $\forall x, y \in \underset{\sim}{A}$ $(x R y \Leftrightarrow G(x) \in G(y))$

<u>Pf</u>: - If G is not 1-1, then
$\{x \in \underset{\sim}{A} : \exists y \in \underset{\sim}{A} \ (x \neq y \wedge G(x) = G(y)\}$
is nonempty.
- Fix an $\underset{\sim}{R}$-minimal x from this set.

- since $\underset{\sim}{R}$ is extensional, there are two cases:

① $\exists z \in \underset{\sim}{A}$ s.t. $z \underset{\sim}{R} x$ and $z \not\underset{\sim}{R} y$
- then since $G(z) \in G(x)$ by def'n of G and $G(x) = G(y)$ there must be $w \underset{\sim}{R} y$ s.t. $G(w) = G(z)$
- then $w \neq z$ by assumption since $z \not\underset{\sim}{R} y$
- this contradicts minimality of x



② $\exists z \in \underset{\sim}{A}$ s.t. $z \underset{\sim}{R} y \wedge z \not\underset{\sim}{R} x$
- mostly symmetric... you try!

↳ hence G is 1-1 ✓

- To see G is an isomorphism:
  - by def'n of G we have $x \underset{\sim}{R} y \Rightarrow G(x) \in G(y)$
    WTS ($\Leftarrow$)
  - so sps $G(x) \in G(y)$
  - $G(y) = \{G(z) : z \underset{\sim}{R} y\}$ by def'n
    hence $G(x) = G(x')$ for some $x' \underset{\sim}{R} y$
  - but then $x = x'$ by 1-1'ness
  $\Rightarrow x R y$ ✓

↳ As a corollary to Lemma, we can finally show every model of ZFC is isomorphic to a transitive model.

- Sps $\langle N, E \rangle$ is a set model of ZFC
  i.e. N is a set, E is a binary rel'n on N, and $\langle N, E \rangle \models ZFC$
- since $\langle N, E \rangle \models$ Extensionality, E is extensional on N
- since $\langle N, E \rangle \models$ Foundation, E is well-founded on N
- for any $x \in N$, $\{y \in N : y E x\}$ is a set since N is itself a set, so E is set-like on N

- Let M be the Mostowski collapse of $\langle N, E \rangle$
- then $\langle M, \in \rangle$ is transitive and isomorphic to $\langle N, E \rangle$
  (in particular $\langle M, \in \rangle \models ZFC$ too)

Relativization

- Now want to figure out what's absolute between (transitive) models of ZFC
- First need to formalize concept of relativization.

Def'n Sps M is a class (possibly a set... gonna stop writing ~'s)

For any formula φ, we define φ^M, the relativization of φ to M, by induction on construction of φ.

- $(x=y)^M$ is $x=y$
- $(x\in y)^M$ is $x\in y$
- $(\varphi\wedge\psi)^M$ is $\varphi^M\wedge\psi^M$
- $(\neg\varphi)^M$ is $\neg\varphi^M$
- $(\exists x\,\varphi)^M$ is $\exists x\,(x\in M\wedge\varphi^M)$

- Note: if M is a set, can take above as literal def'n of φ^M
- If M is a class, defined by a formula Φ, then in last line should really write

$$\exists x\ (\Phi(x)\wedge\varphi^M)$$

- less formally: φ^M is fmla obtained by replacing every "$\exists x$" in φ with "$\exists x \in M$"

- <u>Note</u>: have only defined φ^M for formulas involving $\wedge, \neg$, and $\exists$
- but can get for formulas w/ $\vee, \Rightarrow, \Leftrightarrow, \forall$ from these
- e.g. viewing $\forall x\, \varphi$ as abbreviation for $\neg \exists x \neg \varphi$ we get
  $$(\forall x\, \varphi)^M \text{ is } \neg \exists x\, (x \in M \wedge \neg \varphi^M)$$
- could abbreviate this as $\forall x \in M\, \varphi^M$
- will use such abbreviations freely (easier to read)

- intuition is: if $\varphi(x_1, \ldots, x_n)$ is a fmla then $\varphi^M(x_1, \ldots, x_n)$ asserts "φ is true of $x_1, \ldots, x_n$ within the universe M"

<u>Convention</u>: "φ is true in M" or "$M \models \varphi$" both mean: φ^M can be proved from ZFC (or whatever fragment of ZFC we're currently using)

<u>Ex</u>: the axiom of Extensionality relativized to M can be written:
$$\forall x, y \in M\, (\forall z \in M\, (z \in x \Leftrightarrow z \in y) \Rightarrow x = y)$$