<u>Theorem</u> The following relations and functions were defined in ZF − Powerset by fmlas equiv to Δ_0 fmlas. Hence they are absolute for transitive models of ZF − Powerset.

(a) "x is an ordinal"
(b) "x is a limit ordinal"
(c) "x is a successor ordinal"
(d) "x is a finite ordinal"
(e) ω
(f0) 0 (f1) 1 (f2) 2 ...

<u>Pf</u>: − official def'n of "x is an ordinal" is "x is transitive $\wedge$ x is well-ordered by $\in$"
− but we showed: in transitive models of Foundation "x is well-ordered by $\in$" can be replaced by "x is totally ordered by $\in$"
− already knew "x is transitive" is abs.
− "x is tot. ordered by $\in$" is written $\forall a, b, c \in x$ [... $\in$ is irreflexive, transitive, total...] which is Δ_0.
$\Rightarrow$ "x is an ordinal" is abs.

− informally: "x is transitive and tot. ordered by $\in$" requires only bdd search + verification

- worth noting: "x is well-ordered by $\in$" would on the surface require an unbdd search
- "every non-$\emptyset$ $y \subseteq x$ has a $\in$-least el't" could (in principle) be different between M and V:
  - maybe $\exists$ a subset $y \subseteq x$ in V but not in M that have no $\in$-least el't

↳ so, Foundation doing work here!

(b) equiv to "x is an ordinal $\wedge$ $\forall y \in x \, \exists z \in x \, (z \in y)$" both abs.

(c) equiv to "x is an ordinal and not a limit" which is abs. by (b)

(d) equiv to "x is a successor ordinal containing only 0 and successor ordinals" abs. by (c)

(e) "x is a limit ordinal containing only 0 and successor ordinals" abs by (c), (d)

(Fn) "$x = 0$" is absolute, and then by induction "$\exists y \in x \, (y = n \wedge x = y \cup \{y\})$" is absolute
$\Downarrow$ "$x = n+1$"

<u>Theorem</u> if M is a transitive model of ZF − Powerset, then every finite subset of M is an element of M.

<u>Pf</u>: − Sps $x \subseteq M$, $x = \{y_1, y_2, \ldots, y_n\}$ is a finite subset of M.

− know $\{y_i\} \in M$ $\forall i \leq n$ by absoluteness of pairing

− so by induction $\{y_1, y_2\} \in M$

$$= \{y_1\} \cup \{y_2\}$$

by absoluteness of $\cup$ $\quad \{y_1, y_2, y_3\} \in M$

$$= \{y_1, y_2\} \cup \{y_3\}$$

etc.

$\Rightarrow \{y_1, \ldots, y_n\} \in M$ i.e. $x \in M$ ✓

<u>Theorem</u> The following are absolute for any M satisfying ZF − Powerset

(a) "x is finite"

(b) A^n

(c) $A^{<\omega}$ $\quad \left(= \bigcup_{n \in \omega} A^n\right)$

<u>Pf</u>: − (a) equiv to "$\exists n \in \omega$ and a 1-1 function $f: x \to n$"

− all abs. notions w/ bdd quantifiers except the unbdd "$\exists f$"

− need to check $\exists$ such an $f \in M$ iff $\exists$ such an $f \in V$

- if $\exists f \in M$ then (by absoluteness of notions involved) same f works in V
- for backward direction, notice: if $\exists$ such an $f \in V$ then f is a finite set of ordered pairs $\langle y, k \rangle$ for $y \in x$, $k \leq n$
- all these ordered pairs belong to M ($x \in M$ so $x \subseteq M$, $\omega \subseteq M$, $x \times \omega \in M$, ...)
- hence by prev. lemma $f \in M$ ✓

(b) (c) a little more to think about depending on how we define A^n (best approach - as set of functions from n to A — you try!) ✓

<u>Theorem</u> The following are absolute for any transitive model M of ZF−P

(a) "R well-orders A"

(b) type(A, R)

<u>Pf</u>: - for (a), need to check (R well-orders A)^M iff (R well-orders A) ✓

i.e. R actually well-orders A

- ($\Leftarrow$) is clear since if every non $\emptyset$ $B \subseteq A$ has an R-least elt, then same is true for every such $B \in \underline{M}$

($\Rightarrow$) trick here is to recall: is a theorem of ZF−P that every well-order $\langle A, R\rangle$ is isomorphic to a unique ordinal $\langle \alpha, \in\rangle$ (= type(A,R))

- thus if $(R \text{ well-orders } A)^M$ then there are $f, \alpha \in M$ s.t. "α is an ordinal and f is an order-isomorphism of $\langle A, R\rangle$ w/ $\langle \alpha, \in\rangle$" is true in M
- "α is an ordinal" is absolute and "f is ..." can be checked to be abs. (no unbounded search!)
- hence α really is an ordinal and f really is an isomorphism from $\langle A, R\rangle$ to $\langle \alpha, \in\rangle$
- i.e. R actually well-orders A
- arg also shows: type(A,R) is abs ✓

- Can use theorem to prove the ordinal sum and product are absolute.

<u>Theorem</u> The following are absolute for any transitive model of $ZF - P$.

(a) $\alpha + 1$

(b) $\alpha + \beta$

(c) $\alpha \cdot \beta$

<u>Pf</u> (a) $\alpha + 1$ is $S(\alpha) = \alpha \cup \{\alpha\}$ which we saw is absolute

(b) $\alpha + \beta = \text{type}(\{0\} \times \alpha \cup \{1\} \times \beta, <_{lex})$

everything here absolute:

$\{0\} \times \alpha$, $\{1\} \times \beta$, $\cup$

can check $<_{lex}$ is abs.

type(A, R) abs. by above

(c) you try! ✓

↳ here we used that our official defns of $\alpha + \beta$ and $\alpha \cdot \beta$ were combinatorial

- OTOH we defined α^β by recursion
- to see absoluteness of α^β, let's check that recursion (relative to an absolute relation over an absolute class wrt an absolute function) is absolute.