Absoluteness of classes

- a class A is formally just the formula $\varphi(x)$ defining A
- but we think of A as $\{x : \varphi(x)\}$
  i.e. $\{x \in V : \varphi(x)\}$
- we'll think of A as being <u>absolute</u> (to a model M) if φ is absolute
  i.e. $\varphi^M(x) \Leftrightarrow \varphi(x)$

- we'll write A^M for $\{x \in M : \varphi^M(x)\}$
  = "A relativized to M"
- so: A is absolute for M if $A^M = \{x \in M : \varphi(x)\}$
  $= A \cap M$.

- e.g. if M is a transitive model of ZF-P, then since $\varphi(x) :=$ "x is an ordinal" is abs for M we have $ON^M = ON \cap M$ (i.e. ON is abs. for M)
- OTOH if we let C = class of cardinals $= \{x \in V : \text{"}x \text{ is a cardinal"}\}$ then since "x is a cardinal" is <u>not</u> absolute in general (e.g. if M is a ctbl model of ZFC) we'll usually have $C^M \neq C \cap M$ (though will have $\supseteq$ for c.t.m's M, why?)

- similarly, if R is a (class) relation, formally R is just a formula $\varphi(x,y)$ but we think of R as $\{\langle x,y\rangle \in V\times V : R(x,y)\}$
- we write R^M for $\{\langle x,y\rangle \in M\times M : \varphi^M(x,y)\}$ and say R is absolute (for M) if φ is, i.e. $R^M = R\cap M\times M$

- similarly, if F is a (class) function i.e. F is a class relation defined by $\varphi(x,y)$ and $\forall x \exists! y\, \varphi(x,y)$ is provable, then we say F is an <u>M function</u> if $(\forall x \exists! y\, \varphi(x,y))^M$ is provable and say F is <u>absolute</u> if <u>moreover</u> $F^M = F\restriction M$

- e.g. if $F(x) = x\cup\{x\}$ (i.e. F is defined by the fmla $\varphi(x,y)$ that asserts y is $x\cup\{x\}$) then F is an absolute class function for any transitive model M of (enough of) ZFC, since "$y = x\cup\{x\}$" is absolute and so is the assertion "$\forall x \exists! y\, (y = x\cup\{x\})$"

- whereas: if $F(x) = |x|$ (i.e. F is defined by the formula $\varphi(x,y)$ that asserts "y is the least ordinal for

which there is a bijection $f: y \to x$") then F is <u>not</u> absolute (e.g. a c.t.m. M may compute $F(x)$ very differently than V, for which always $F(x) \leq \omega$)

- Following theorem says: if everything in sight is absolute <u>and</u> the relation R is closed downward in M, then recursion on R is absolute.

<u>Theorem</u>: Sps R is a relation that is w.f. and set-like on A and $F: A \times V \to V$ is a function.

Let $G: A \to V$ be the function given by the recursion theorem, i.e. the unique function s.t.

$$\forall x \in A \; [G(x) = F(x, G \restriction pred_R(x))]$$

Sps M is a transitive model of ZF-P and moreover

① F is absolute for M

② R and A are absolute for M, $(R \text{ is set-like on } A)^M$ <u>and</u>

$$\forall x \in M \; (pred_R(x) \subseteq M)$$

downward closure

<u>then</u> G is absolute for M

Pf: omitted.

↳ approximately: hypothesis give that tree diagram for R^M on A^M below a given $x \in A^M$ (this is downward closure) is the same computed in M as in V

↳ hence, since $G(x)$ depends only on $G(y)$ for y below G (and F which is absolute) we have $G^M(x) = G(x)$ ✓

Theorem The following are absolute for any transitive model M of $ZF - P$.

(a) α^β

(b) $\mathrm{rank}(x)$ (we use the recursive def'n $\mathrm{rank}(x) = \sup\{\mathrm{rank}(y)+1 : y \in x\}$)

(c) $\mathrm{trcl}(x)$

Pf: α^β is defined by an absolute recursion on β

$\mathrm{rank}(x)$ is defined by an absolute recursion on x

$\mathrm{trcl}(x)$ is defined by first an absolute recursion on ω to define each $\bigcup^n x$, then taking a union over these.

- We close w/ a word on the absoluteness of powersets and the levels $R(\alpha)$ of the WF hierarchy.

- Sps now M is a transitive model of ZF (so $M \models$ Powerset)
- $P(x)$ (i.e. the class function that assigns to each x its powerset $P(x)$) will typically not be absolute
  (if M is ctbl then e.g. $P^M(x) \neq P(x)$)

- however: since "$x \subseteq y$" is absolute we have $P^M(x) = M \cap P(x)$

- likewise, for an ordinal $\alpha \in M$, we typically have $R(\alpha)^M \neq R(\alpha)$

- however, since "$y = \text{rank}(x)$" is abs. and $R(\alpha) = \{x : \text{rank}(x) < \alpha\}$ we do have $R(\alpha)^M = \{x \in M : \text{rank}(x) < \alpha\}$
  $= R(\alpha) \cap M$

Forcing

- Recall idea of forcing:
  - start w/ a ctm $M \models ZFC$
    (sweeping some Gödel issues under the rug: really we fix a ctm M of a large enough finite fragment of ZFC to carry out "all arguments we're going to do.")
  - Given a p.o. $\mathbb{P} \in M$, we find (from the outside) a generic filter $G \subseteq \mathbb{P}$ i.e. $G \cap D \neq \emptyset$ $\forall$ dense subsets $D \subseteq \mathbb{P}$ s.t. $D \in M$
  - G gives a way (how? stay tuned) to construct a new c.t.m $M[G]$
  - $M[G]$ is the minimal ctm $N \models ZFC$ s.t. $M \subseteq N$ and $G \in N$
  - what statements beyond ZFC that hold in $M[G]$ (e.g. CH, $\neg$CH, SH, $\neg$SH, $\diamondsuit$, MA, etc.) depend on $\mathbb{P}$!

Generic Extensions

- We modify our previous def'n of "partial order" now insisting on a distinguished largest el't

  convenient for forcing formalism.

<u>Def'n</u> A <u>partial order</u> (p.o.) is a triple $\langle \mathbb{P}, \leq, \mathbb{1} \rangle$ where $\leq$ is a reflexive, transitive rel'n on $\mathbb{P}$ and $\mathbb{1}$ is a largest el't of $\mathbb{P}$ (i.e. $\forall p \in \mathbb{P}, p \leq \mathbb{1}$)

- <u>note</u>: $\mathbb{1}$ is a <u>distinguished</u> largest el't
- since $\leq$ may not be antisymmetric, maybe $\exists p \in \mathbb{P}, p \neq \mathbb{1}$ s.t. $p \leq \mathbb{1} \leq p$.
- then p also a largest el't.

- we'll write $\mathbb{P} \in M$ to abbreviate $\langle \mathbb{P}, \leq, \mathbb{1} \rangle \in M$ (which, by transitivity yields $\mathbb{P} \in M$, $\leq \in M$, $\mathbb{1} \in M$, $\mathbb{P} \subseteq M$, $\leq \subseteq M$)

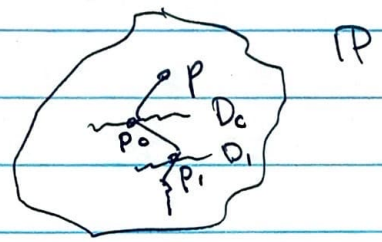
<u>Def'n</u> Sps M is a ctm, $\mathbb{P} \in M$ is a p.o. A filter $G \subseteq \mathbb{P}$ is <u>$\mathbb{P}$-generic over M</u> if for every dense subset $D \subseteq \mathbb{P}$ s.t. $D \in M$ we have $G \cap D \neq \emptyset$.

↳ we can always find generic filters over ctm's.

<u>Lemma</u> Sps M is a ctm, $\mathbb{P} \in M$ is a p.o, and $p \in \mathbb{P}$. There is a filter $G \subseteq \mathbb{P}$ that is $\mathbb{P}$-generic over M with $p \in G$.

Pf - similar to proof of MA(ω) from last quarter

- enumerate (in V) all dense subsets $D \subseteq \mathbb{P}$ s.t. $D \in M$: $D_0, D_1, D_2, \ldots$
  (only ctbly many - M itself is ctbl!)
- since D_0 is dense, can find $p_0 \leq p$ s.t. $p_0 \in D_0$
- then find $p_1 \leq p_0$ s.t. $p_1 \in D_1$, etc.
  $p_2 \leq p_1$ $p_2 \in D_2$
  $\vdots$



- let G = filter generated by $\{p_0, p_1, p_2, \ldots\}$ (i.e. upward closure of this set in $\mathbb{P}$)
- then G is $\mathbb{P}$-generic over M by construction: $G \cap D_i \neq \emptyset$ ✓

<u>Note on absolutenes:</u>

- we're implicitly working in V above.
- i.e. "$\mathbb{P} \in M$ is a p.o." means "$\langle \mathbb{P}, \leq, \mathbb{1} \rangle$ is a p.o." (in V) and also $\langle \mathbb{P}, \leq, \mathbb{1} \rangle \in M$
- but "$\mathbb{P}$ is a p.o." is absolute for trans. models of ZFC
  (why? think it thru: just asserts $\leq$ is refl., trans. on $\mathbb{P}$, $\mathbb{1}$ is max'l in $\mathbb{P}$, all bounded quantifier statements...)