<u>Theorem</u> (Kleene normal form theorem for r.e. relations)

There is a p.r. relation $T(e,x,c)$ s.t. for every r.e. rel'n $R \subseteq \mathbb{N}^n$, there is an e s.t.

$$R(\vec{x}) \text{ iff } \exists c\, T(e, \langle \vec{x} \rangle, c)$$

<u>Pf</u>. - Sps R is r.e.

- fix f partial recursive s.t. $R = \mathrm{dom}(f)$
- there is e s.t. $f(\vec{x}) = U(\mu c T(e, \langle \vec{x} \rangle, c))$
- so: $R(\vec{x})$ iff $f(\vec{x})\downarrow$

  iff $\exists c. T(e, \langle \vec{x} \rangle, c)$ ✓

<u>Universal relations</u>:

~~[illegible]~~

~~[illegible]~~, if $\mathcal{R}$ is a class of relations on $\mathbb{N}$, let $\mathcal{R}_n$ = n-ary rel'ns in $\mathcal{R}$

<u>Def'n</u> $W \subseteq \mathbb{N}^{n+1}$ is universal for $\mathcal{R}_n$ if $\mathcal{R}_n = \{W_e : e \in \mathbb{N}\}$

where: $W_e(\vec{x}) = W(e, \vec{x})$

<u>Def'n</u> $\mathcal{R}$ has <u>universal relations</u> (or the <u>enumeration property</u>) if for each $n \in \mathbb{N}$, there is a $W \in \mathcal{R}_{n+1}$ that is universal for $\mathcal{R}_n$.

<u>Prop'n</u> if $\mathcal{R}$ is class of recursive relations (or p.r. relations) then $\mathcal{R}$ does not have enumeration property.

<u>Pf</u>: - diagonalize; key point is these classes are closed under $\neg$, complementation
- more precisely: sps $W \subseteq \mathbb{N}^2$ is universal for $\mathcal{R}$,
- define $K(e)$ iff $W(e,e)$
- then K also recursive
- also: $\neg K$ recursive
- so there is e_0 s.t. $\neg K(n)$ iff $W(e_0,n)$
- but then $\neg K(e_0)$ iff $W(e_0,e_0)$ (by above)
iff $\neg W(e_0,e_0)$ (def'n of K)
- contradiction ✓

<u>Theorem</u> (Kleene enumeration theorem) the class $\mathcal{R}$ of r.e. relations has the enumeration property

"x is in dom(φ_e)"

<u>Pf</u>: - define $W(e,x)$ iff $\exists c\, T(e,x,c)$
- W universal for $\mathcal{R}$, by above ✓
- can code higher arity relations as unary relations (you try) ✓

<u>Corollary</u>: (i.) r.e. rel'ns <u>not</u> closed under $\neg$ (complements), $\forall$ (unbdd $\forall$)

(2.) There is an r.e. relation which is not recursive

Pf: – For $W(e,x)$ the universal r.e. relation, define $K(e)$ to be $W(e,e)$
– also r.e. by prev. theorem (substituting W in both coords by identity function)
– but $\neg K$ not r.e. (otherwise could do diagonal arg and prove r.e. doesn't have enum. prop)
– hence r.e. not closed under $\neg$
– also follows: K is not recursive (o/w $\neg K$ is recursive $\Rightarrow$ r.e.) ✓

Decidability + the halting problem:

– Sps M is a TM on $A \cup \{,\}$ (alphabet w/ comma)
– Consider the relation
$C_M(w)$ iff M halts on input $w \in A^*$

Halting problem for M: is C_M recursive?
("is there an algorithm that determines if M halts on a given input?")

Prop'n There is a TM M_0 on $\{1, ,\}$ for which C_{M_0} is not recursive

<u>Pf</u>: - Let $\varphi(e,x) := U(\mu c\, T(e,x,c))$
be an univ. recursive function
- define $\varphi'(z) := \varphi((z)_0, (z)_1)$
- φ' also recursive
- Let M_0 be a TM computing φ'
- recall $K(e)$ from above:

$K(e)$ iff $W(e,e)$
iff $\exists c\, T(e,e,c)$
iff $\varphi(e,e)\downarrow$
iff $\varphi'(\langle e,e\rangle)\downarrow$
iff $C_{M_0}(\langle e,e\rangle)$

- we showed K not recursive
- $\Rightarrow$ C_{M_0} not either
↳ if it were could define K by
$K(e)$ iff $\underbrace{C_{M_0}(\langle e,e\rangle)}$
would be recursive since obtained from
$C_M(w)$ by total rec. sub'n
↳ contradiction ✓

Structure of this proof

- show: $K(e)$ iff $C_{M_0}(\langle e,e\rangle)$
- map $e \to \langle e,e\rangle$ total recursive
$\Rightarrow$ if C_{M_0} recursive, K is too
<u>Pf</u>: $\chi_K(e)$ iff $\chi_{C_{M_0}}(\langle e,e\rangle)$
↖ recursive

but K not recursive $\Rightarrow$ C_{M_0} not either.

- illustrates a <u>reduction</u> of problem "membership in C_{M_0}" to "membership in K"
- In general: if A not recursive and want to show B not recursive, can do so by finding total rec. f s.t.

$$A(x) \text{ iff } B(f(x))$$

- on HW6: you'll show a variation of halting prob not recursive
- may help to reduce to Halting problem above!

<u>Effective enumerations and the s-m-n theorem</u>

<u>Def'n</u> an enumeration $\psi_0, \psi_1, \ldots$ of the (unary) recursive functions is <u>effective</u> if the (binary) function $\psi(e,x) := \psi_e(x)$ is recursive.

- our univ. rec. function $\varphi(e,x)$ is effective
- $\varphi(e,x) = \varphi_e(x) = U(\mu c\, T(e,x,c))$
- <u>recall</u>: - if e not the code of a TM on $\{1, \}$ then φ_e is empty function
  - if e codes such a TM M, φ_e is unary function computed by M

i.e. $\varphi_e = \varphi_M^1$

- then defined univ n-ary rec. function by:

$$\varphi^n(e,\vec{x}) = \varphi(e,\langle\vec{x}\rangle) = \varphi_e(\langle\vec{x}\rangle)$$

- similarly, if $\psi(e,x)$ is another enumeration of unary rec. functions, we'll write $\psi^n(e,\vec{x})$ to mean $\psi(e,\langle\vec{x}\rangle) := \psi_e(\langle\vec{x}\rangle)$
- sometimes we'll be informal and drop superscripts: write $\varphi_e(\vec{x})$ $\psi_e(\vec{x})$ ... always understood these mean $\varphi_e(\langle\vec{x}\rangle)$, $\psi_e(\langle\vec{x}\rangle)$.

<u>Next</u>: want to define what it means for a functional <u>operator</u> (e.g. composition, minimalization) to be recursive.

- let Ω_k = set of partial functions $f: N^k \to N$
- let $\Omega = \bigcup_{k\in N} \Omega_k$

<u>Def'n</u> An <u>operator</u> of <u>type $(t;k_1,\ldots,k_m;k)$</u> (where $t\geq 0$, k_i and $k\geq 1$) is a map

$$\Phi: N^t \times \Omega_{k_1} \times \cdots \times \Omega_{k_m} \to \Omega_k$$

↪ idea is: given $f_1,\ldots,f_m$ (of arities $k_1,\ldots,k_m$) and some parameters $x_1,\ldots,x_t \in N$ Φ constructs a new function f (of arity k)

↪ write $\Phi(x_1,\ldots,x_t,f_1,\ldots,f_m) = f$

Ex's ① Composition operator:

- fix $m, k \geq 1$; define $\Phi_c^{m,k}$ ← "composition"

to be the operator of type

$(0; m, \underbrace{k, k, \ldots, k}_{m\text{-times}}; k)$

defined by $\Phi_c^{m,k}(f, g_1, \ldots, g_m) = h =$ comp'n of f w/ g_i's $= f(g_1, \ldots, g_m)$

(i.e. $h(\vec{x}) = f(g_1(\vec{x}), \ldots, g_m(\vec{x}))$)

② Primitive recursion operator: Φ_p^k

<u>for arity $k > 1$</u> has type $(0; k-1, k+1; k)$

defined by $\Phi_p^k(g_1, g_2) = h =$ func. obtained from g_1, g_2 by p.r.

i.e. $h(0, \vec{x}) = g_1(\vec{x})$, $h(i+1, \vec{x}) = g_2(h(i,\vec{x}), i, \vec{x})$

<u>for arity $k = 1$</u> has type $(0; 2; 1)$

given $w_0 \in \mathbb{N}$, denote as $\Phi_p^{w_0}$, defined by $\Phi_p^{w_0}(g) = h$ where $h(0) = w_0$

$h(i+1) = g(h(i), i)$

③ minimalization operator Φ_μ^k:

for $k \geq 1$ has type $(0; k+1, k)$

$\Phi_\mu^k(g) = h$ where $h(\vec{x}) = \mu t\,[g(t, \vec{x}) = 0]$

④ Fixing operator $\Phi_F^{m,n}$:

for $m, n \geq 1$ has type $(m; m+n; n)$

defined by

$\Phi_F^{m,n}(x_1, \ldots, x_m, g) = h$ where

$h(y_1, \ldots, y_n) = g(x_1, \ldots, x_m, y_1, \ldots, y_n)$

(we fixed first m args for g)

Def'n The recursive functions are **uniformly closed** (modulo an enumeration ψ) under an operator $\Phi(\vec{x}, f_1, \ldots, f_m)$ if there is a total recursive function $F(\vec{x}, e_1, \ldots, e_m)$ s.t.

$$\Phi(\vec{x}, \psi_{e_1}^{k_1}, \ldots, \psi_{e_m}^{k_m}) = \psi^{k}_{F(\vec{x}, e_1, \ldots, e_m)}$$

How to read: Φ a way of constructing $f = \Phi(\vec{x}, f_1, \ldots, f_m)$ out of the rec. functions $f_1, f_2, \ldots, f_m$ (and params $\vec{x}$)

- given codes $e_1, \ldots, e_m$ for these functions (so $f_i = \psi_{e_i}^{k_i}$) (... of arity k_i)

F returns code for $\Phi(\vec{x}, f_1, \ldots, f_m)$

- so "uniformly closed" is right notion of Φ being a recursive operator (i.e. a "recursive way" of constructing new f from f_i)

- we'll show our enumeration $\varphi_e(x) = U(\mu c\, T(e, x, c))$ has a stronger property than just being effective (i.e. recursive)

Def'n an effective enumeration ψ is **acceptable** if there is a total recursive function $S(e, x)$ s.t.

$$\psi_e(x, y) = \psi_{S(e,x)}(y)$$

(i.e. $\psi(e, x, y) = \psi(S(e, x), y)$

where $\psi(e, x, y) = \psi^2(e, x, y) = \psi(e, \langle x, y\rangle)$
$= \psi_e(\langle x, y\rangle)$)