Def'n An operator $\Phi(\vec{x},\vec{f})$ is **continuous** if

$\Phi(\vec{x},\vec{f})(\vec{y}) = z$ if and only if
for every f_i, $\exists$ a finite $\pi_i \subseteq f_i$
s.t. $\Phi(\vec{x},\vec{\pi})(\vec{y}) = z$

How to read: for a given input $\vec{y}$
if $\Phi(\vec{x},\vec{f})(\vec{y})\downarrow$
need only a finite amount of info
about the f_i's to compute $\Phi(\vec{x},\vec{f})(\vec{y})$

More precisely: if Φ cts, then for any
$\vec{f} = f_1, \dots, f_m$

① if $\Phi(\vec{x},\vec{f})(\vec{y})$ is defined (say $= z$)
can find finite approx's $\pi_i \subseteq f_i$ s.t.
$\Phi(\vec{x},\vec{\pi})(\vec{y}) = z$ too

② if π_i, $1 \le i \le m$ are finite and
$\Phi(\vec{x},\vec{\pi})(\vec{y})\downarrow$ (say $= z$)
then for any $g_1, \dots, g_m$ s.t. $\pi_i \subseteq g_i$
must have $\Phi(\vec{x},\vec{g})(\vec{y}) = z$ too
(incl. $\vec{f}$, if $f_i \supseteq \pi_i$)

→ We'll show: "Φ is cts iff effective" in precise sense
Myhill–Shepherdson Theorem

<u>Def'n</u> Φ is <u>monotone</u> if $f_i \subseteq g_i$ $1 \le i \le m$
$\Rightarrow \Phi(\vec{x}, \vec{f}) \subseteq \Phi(\vec{x}, \vec{g})$

i.e. if g_i's $= f_i$'s (except maybe more outputs) then $\Phi(\vec{x}, \vec{g}) = \Phi(\vec{x}, \vec{f})$ (except maybe more outputs)

<u>Note 1</u>: Continuity implies monotonicity

<u>why</u>: if Φ cts, $f_i \subseteq g_i$ $\forall i$, then by 1 above can find $\pi_i \subseteq f_i$ s.t. (for given $\vec{x}, \vec{y}$ s.t. $\Phi(\vec{x}, \vec{f})(\vec{y}) = z$) s.t. $\Phi(\vec{x}, \vec{\pi})(\vec{y}) = z$ too. But then $\Phi(\vec{x}, \vec{g})(\vec{y}) = z$ by 2.

<u>Note 2</u>: our def'ns of "continuous" and "monotone" made wrt a global operator Φ, can also consider restrictions of these def'ns when inputs $\vec{f}$ are recursive.

<u>Note 3</u>: there is a natural topology on Ω_K s.t. our def'n of "continuous" really is cts in this topology.

- Following theorem characterizes the effective operators Φ
  (recall: Φ is effective if F_Φ recursive wrt acceptable enum)
- Thm roughly says "Φ is effective iff cts on the recursive functions"

- Let $\pi(m,x) = \pi^m_x$ be a recursive enumeration of the m-ary finite functions
- i.e. π recursive and for a fixed m, π^m_x enumerates fin m-ary functions as x varies over $\mathbb{N}$.

<u>Theorem</u> (Myhill – Shepherdson)
(think of as "operator analogue" of Rice – Shapiro)

Sps $\Phi(\vec{x}, \vec{f})$ is an operator of type $(n; k_1, .., k_m; k)$. TFAE:

① Φ is an effective operator
② Φ restricted to the rec. functions is continuous, and the function f_Φ defined by
$$f_\Phi(\vec{x}, \vec{t}, \vec{z}) = \Phi(\vec{x}, \pi^{k_1}_{t_1}, .., \pi^{k_m}_{t_m})(\vec{z})$$
is recursive
③ There is a (uniquely determined) effective operator $\overline{\Phi}$ which is cts (i.e. globally, i.e. on all functions not just recursive) s.t. $\overline{\Phi}$ agrees w/ Φ on the rec. functions

<u>Pf</u>: (sketch) For simplicity we assume Φ of type (0;1;1)

both unary, no numerical parameters

i.e. $\Phi(f) = g$

$\underline{\underline{① \Rightarrow ②}}$. Recall: $\Phi(f)$ effective iff functional $\Phi^*(f,z)$ is recursive (really mean: the encoded functional, on recursive inputs f) iff graph of functional $G_\Phi(f,z,w)$ is r.e. (really mean: encoded graph)

- So sps Φ effective
- By Rice-Shapiro (really: ①⇔② part of R-S) since G_Φ r.e. have:
  (i) $\{(\pi,z,w) : \pi$ finite function $\wedge\ G_\Phi(\pi,z,w)\}$ is r.e. (really mean: $\{(t,z,w) : G_\Phi(\pi_t,z,w)\}$)
  i.e. graph of f_Φ is r.e.
  which gives f_Φ is recursive (second part of ②)
  (ii) for f recursive, have
  $G_\Phi(f,z,w)$ iff $\exists$ finite $\pi \subseteq f$ s.t. $G_\Phi(\pi,z,w)$ iff $\exists$ finite $\pi \subseteq f$ s.t. $\Phi(\pi)(z) = w$, which gives continuity of Φ (i.e. first part of ②) ✓

$\underline{② \Rightarrow ①}$ Same in reverse:
- Sps Φ is cts on rec. functions and f_Φ is recursive; WTS Φ is effective, i.e. G_Φ is r.e.
- Have:
  $G_\Phi(f,z,w)$ iff $\Phi(f)(z) = w$
  iff (by continuity) $\exists$ fin. $\pi \subseteq f$, $\Phi(\pi)(z) = w$
  iff $\exists \pi \subseteq f$ $G_\Phi(\pi,z,w)$

iff $\exists t\,(\pi_t \subseteq f \wedge G_{f_\Phi}(t,z,w))$

graph of f_Φ

this is r.e. since f_Φ rec. and hence G_{f_Φ} r.e. (so of form $\exists t$ (r.e.) — can you make precise?)

- thus G_Φ r.e. ✓

③ ⇒ ① trivial (you try)

<u>② ⇒ ③</u>. Let Φ_0 = restriction of Φ to rec functions; we're assuming eff on rec. functions

- want to extend to (globally) eff $\overline{\Phi}$
- will define graph
- only one way if we want continuity:

$$\overline{\Phi}(f)(z) = w \text{ iff } \exists \text{ finite } \pi \subseteq f\ \overline{\Phi}(\pi)(z) = w$$

- this is circular as written, but since we want $\overline{\Phi} = \Phi_0$ on rec. functions, and all finite functions are rec., can define:

$$\overline{\Phi}(f)(z) = w \text{ iff } \exists \text{ fin. } \pi \subseteq f \text{ s.t. } \Phi_0(\pi)(z) = w$$

- is this well-defined? i.e. is $\overline{\Phi}(f) = g$ a function for all f?
- sps $\exists$ finite $\pi_0, \pi_1 \subseteq f$ s.t. $\Phi_0(\pi_0)(z) = w_0$

$\Phi_0(\pi_1)(z) = w_1$

- then $\pi_0 \cup \pi_1$ still fin. subfunction of f
- by monotonicity of Φ_0 have both

$\Phi_0(\pi_0 \cup \pi_1)(z) = \Phi_0(\pi_0)(z)$ and $\Phi_0(\pi_0 \cup \pi_1)(z)$

$= \Phi_0(\pi_1)(z)$

- i.e. $w_0 = w$,
- so this relation really is a graph relation for an operator $\overline{\Phi}(f) = g$ (i.e. spits out a function)

<u>Now</u>: $\overline{\Phi}$ restricted to rec. functions is Φ_0 (why?)
- which is continuous and hence effective (by ② ⇒ ① arg above)
- hence $\overline{\Phi}$ effective (effectiveness only depends on behavior on rec. functions)
- uniquely determined since determined by restriction to fin. functions
- continuous by definition

(in fact: above arg shows if Φ is any operator that is monotone on finite functions, then Φ has unique cts extension to all functions) ✓

<u>Def'n</u> A <u>continuous</u> <u>effective</u> <u>operator</u> is an effective operator that is (globally) continuous

↳ ② of M-S says any effective operator is cts on rec. functions
↳ ③ implies in many cases can assume cts everywhere (replace Φ w/ $\overline{\Phi}$)

want input arity = output arity ... going to iterate

Theorem (First recursion theorem)
Sps $\Phi : \Omega_m \to \Omega_m$ is a continuous effective operator
Then Φ has a least (wrt inclusion) fixed point $f = \Phi(f)$, which is recursive

$\hookrightarrow$ Before proving, we'll prove following "abstract" version (no recursive hypotheses)

Theorem Sps $\Phi : \Omega_m \to \Omega_m$ is a continuous operator. Then Φ has a unique least fixed point f
(i.e. $\Phi(f) = f$ and if $\Phi(g) = g$ then $g \supseteq f$)
(in particular, if f is total, is unique fixed point)
We write f_Φ for f.

Pf: Define inductively
$f_0 = \phi$
$f_{n+1} = \Phi(f_n)$ $\leftarrow$ makes sense $\Phi : \Omega m \to \Omega m$

- By monotonicity of Φ:
$\phi \subseteq \Phi(\phi) \subseteq \Phi(\Phi(\phi)) \subseteq \ldots$
i.e. $f_0 \subseteq f_1 \subseteq f_2 \subseteq \ldots$

- define $f_\Phi = \bigcup_{n \in \mathbb{N}} f_n$, is a function by

- We show $\Phi(f_\Phi) = f_\Phi$ by double containment

- for each n, $f_n \subseteq f_{n+1} = \Phi(f_n)$
  $\subseteq \Phi(f_\Phi)$ (monotone)
  $\Rightarrow \bigcup f_n \subseteq \Phi(f_\Phi)$ i.e. $f_\Phi \subseteq \Phi(f_\Phi)$

- Now we show $\Phi(f_\Phi) \subseteq f_\Phi$
- Sps $\Phi(f_\Phi)$ defined @ $\vec{x}$, say
  $\Phi(f_\Phi)(\vec{x}) = y$
- By continuity $\exists$ finite $\pi \subseteq f_\Phi$ s.t.
  $\Phi(\pi)(\vec{x}) = y$
- Since π finite, $\exists n$ s.t. $\pi \subseteq f_n$
- by monotonicity, $\Phi(\pi) \subseteq \Phi(f_n)$
  so $\Phi(f_n)(\vec{x}) = y$ too, i.e. $f_{n+1}(\vec{x}) = y$
  which gives $f_\Phi(\vec{x}) = y$
- so $\Phi(f_\Phi) \subseteq f_\Phi$

- thus $\Phi(f_\Phi) = f_\Phi$, as claimed

- new: if $\Phi(g) = g$ is another fixed point (or even $\Phi(g) \subseteq g$)
  then by induction (using monotonicity)
  $\Phi(f_n) \subseteq g$ for all n
  i.e. $f_{n+1} \subseteq g$ $\forall n$
  $\Rightarrow f_\Phi \subseteq g$ ✓

↪ we'll write Φ^n for f_n
and Φ^∞ for f_Φ
from proof.