11 For $x, y \in A^*$, define $Ex'(x,y) := x^{|y|} = ~~[scribbled out]~~ xx\cdots x$ ($|y|$ times)

(note $x^0 = \sigma_1$)

Then Ex' is pr

pf. $Ex'(x,0) = \sigma_1$

$Ex'(x, y\sigma_j) = con_2(Ex'(x,y), x)$

12 define $Ex(x) = x^{|x|}$

Then Ex is p.r. (why?)

Recall: - $A = \{\sigma_1, \ldots, \sigma_k\}$ is a finite alphabet; A^* = all (finite) words on A.

- a function

$$f : (A^*)^n \to A^*$$

is pr if formed from initial functions (erase E, successors S_j $1 \le j \le k$, projections P_i^n) using composition and prim. recursion

Q: what does it mean for a set, or relation, to be p.r.?

Def'n Given a set X, an n-ary relation on X is a subset $R \subseteq X^n$

↳ e.g. can think of $\le$ on $\mathbb{N}$ as $\{(n,m) : n \le m\} \subseteq \mathbb{N}^2$

– write $R(x_1, \ldots, x_n)$ if $(x_1, \ldots, x_n) \in R$
"$R(\vec{x})$" "$\vec{x} \in R$"

– Given n-ary relations R, P can write: $\neg R$ for $X^n \setminus R$
$P \wedge R$ for $P \cap R$
$P \vee R$ for $P \cup R$

– we'll say a rel'n R on $(A^*)^n$ is p.r. if its characteristic function is p.r.

<u>Def'n</u> Given a relation $R \subseteq (A^*)^n$ define $\chi_R : (A^*)^n \to A^*$ by:

$$\chi_R(w_1, \ldots, w_n) = \begin{cases} 0 & \text{if } R(w_1, \ldots, w_n) \\ \sigma_1 & \text{if not} \end{cases}$$

opp. of usual convention
$\chi_R(\vec{w}) = 0$ means $\vec{w} \in R$
$= 1$ $\vec{w} \notin R$

<u>Def'n</u> a rel'n $R \subseteq (A^*)$ is <u>primitive recursive</u> if χ_R is p.r.

<u>Prop'n</u> ① if P, Q are p.r. relations on $(A^*)^n$, then so are $\neg P$, $P \wedge Q$, $P \vee Q$.

<u>Pf</u>: $\chi_{\neg P}(\vec{x}) = Lev(\chi_P(\vec{x}))$

pr (p.r. by comp'n)

$$\chi_{P\wedge Q}(\vec{x}) = \text{lev}\left(\text{con}_2\left(\chi_P(\vec{x}), \chi_Q(\vec{x})\right)\right)$$

pr by comp'n

now notice: $P \vee Q(\vec{x})$ iff $\neg(\neg P(\vec{x}) \wedge \neg Q(\vec{x}))$

(DeMorgan's)

$\Rightarrow$ $\chi_{P\vee Q}$ is p.r. using above + comp'n ✓

② Sps $P(x_1, \ldots, x_m)$ is m-ary p.r. relation and $f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)$ are p.r. n-ary functions

define $Q(x_1, \ldots, x_n)$ iff

$P(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$

then: Q is p.r.

<u>Pf</u>: $\chi_Q(\vec{x}) = \chi_P(f_1(\vec{x}), \ldots, f_m(\vec{x}))$

<u>e.g.</u> if "$a \leq b$" is p.r. and "+" and "·" are p.r. then so is "$(x+y) \leq x \cdot y$"

③ ("piecewise p.r. is p.r.")

<u>If</u> $P_1, \ldots, P_m$ are <u>disjoint</u> p.r. relations (i.e. $P_i(\vec{x}) \Rightarrow \neg P_j(\vec{x})$ for $i \neq j$)

<u>and</u> $f_1, \ldots, f_m, f_{m+1}$ are p.r. functions

<u>then</u> the function

$$f(\vec{x}) := \begin{cases} f_i(\vec{x}) & \text{if } P_i(\vec{x}) \\ f_{m+1}(\vec{x}) & \text{if none true} \end{cases}$$

is p.r.

<u>Pf</u>: first define

$$P_{m+1} = \neg(P_1 \vee \cdots \vee P_m)$$

... is p.r. by ©

so can think of $f(\vec{x}) = f_i(\vec{x})$

if $P_i(\vec{x})$ ... now for $1 \leq i \leq m+1$

Let's check f p.r.

first consider function $\otimes$:

$$w \otimes v := w \quad \text{if } v = 0$$
$$= 0 \quad \text{if } v \neq 0$$

is p.r. since:

$$w \otimes 0 = w = P^1_1(w)$$
$$v \otimes x\sigma_j = 0 = E(P^3_1(w \otimes x, w, x))$$

now observe: $f(\vec{x}) = \text{con}_{m+1}(f_1(\vec{x}) \otimes \chi_{P}(\vec{x}), \ldots, f_{m+1}(\vec{x}) \otimes \chi'_{P_{m+1}}(\vec{x}))$

which is p.r. ✓

④ <u>Def'n</u> given $y, z \in A^*$ we write $y|z$

"$z = yw$" to mean "y is an initial segment of z"

↪ in case $A = \{a\}$, $A^* = \mathbb{N}$, think of

$y|z$ as "$y \leq z$"

↪ $y = 0$ is initial in <u>every</u> z

<u>Prop'n</u> (bdd quantification is p.r.)

<u>If</u> $P(y, \vec{x})$ is a p.r. relation, then:

$$P_1(z, \vec{x}) := \exists y|z \; P(y, \vec{x}) \quad \text{is p.r.}$$

and

$$P_2(z, \vec{x}) := \forall y|z \; P(y, \vec{x}) \quad \text{is p.r.}$$

Pf. observe $P_1(z,\vec{x})$ iff $\neg \forall y \mid z \neg P(y,\vec{x})$ iff $\neg(\neg P)_2(z,\vec{x})$

∴ so suffices to check P_2 is p.r.

can defin χ_{P_2} w/ recursion:

$$\chi_{P_2}(0,\vec{x}) = \chi_P(0,\vec{x})$$

↳ only init. seg. of 0 is 0

$$\chi_{P_2}(z\sigma_j,\vec{x}) = \text{lor}(\text{con}_2(\chi_{P_2}(z,\vec{x}), \chi_P(z\sigma_j,\vec{x})))$$

which is p.r ✓

(5) (bdd minimalization)

Sps $P(y,\vec{x})$ is a p.r. relation

Then the function

$$f(z,\vec{x}) := \mu y \mid z\, P(y,\vec{x})$$

"least integer $\mathcal{F}z$ s.t. $P(y)$"

write $:=$ shortest init seg y of z s.t. $P(y,\vec{x})$ (if exists)

$= z\sigma_1$ (if no such y)

is p.r.

Pf: Can define f by recursion!

$$f(0,\vec{x}) = \chi_P(0,\vec{x})$$

↳ remember... χ_P outputs σ_1 if false

$f(z\sigma_j,\vec{x}) = f(z,\vec{x})$ if $\exists y \mid z\, P(y,\vec{x})$ ← P_1

$= z\sigma_j$ if $\neg(\ldots) \wedge P(z\sigma_j,\vec{x})$ ← P_2

$= z\sigma_j\sigma_1$ if neither ← P_3

f_1 f_2 f_3

we're using a piecewise defin of f here! Notice P_1, P_2, P_3 are p.r. and disjoin

f_1, f_2, f_3 p.r.

more explicitly: $f_1(f(z,\vec{x}), z, \vec{x}) = f(z,\vec{x})$

$f_2(\ldots) = z\sigma_j$

$f_3(\ldots) = z\sigma_j\sigma_i$

all p.r. (why?) ✓

Ex's (1) the equality relation $x=y$ is p.r. (i.e. the 2-ary rel'n $R \subseteq (A^*)^2$ defined by $R(x,y)$ iff $x=y$)

Pf: Let $\mathrm{First}(x) :=$ first symbol in x ($=0$ if $x=0$)

↳ is p.r. since $= x - (x - \sigma_1)$

$\mathrm{Last}(x) :=$ last symbol in x

↳ also p.r. since $= \mathrm{First}(\mathrm{Rev}(x))$

Note: $x=y$ iff $\forall z|x\ \mathrm{First}(x-z) = \mathrm{First}(y-z)$

(if y shorter or longer, disagree)

So: suffices to check the rel'n

$R(x,y)$ iff $\mathrm{First}(x) = \mathrm{First}(y)$ is pr

($(0,0) \in R$ too.)

since then above rel'n is p.r. by above results.

We check: $R'(x,y)$ iff $\mathrm{Last}(x) = \mathrm{Last}(y)$

is pr, cause then

$R(x,y)$ iff $R'(\mathrm{Rev}(x), \mathrm{Rev}(y))$

can define $\chi_{R'}$ by recurs'n:

$$\chi_{R'}(0, y) = \text{Lev}(y)$$
$$\chi_{R'}(x\sigma_j, y) = \text{End}_j(y)$$

recall: $= 0$ if y ends in σ_j, $= 1$ o/w

done ✓

② $x|y$ is pr (i.e. the rel'n $R(x,y)$ iff x initial seg of y)

Pf: $x|y$ iff $\exists z|y\ (x = z)$
pr; pr by ①

③ Def'n: we write $x \subset y$ if x occurs in y i.e. $y = wxv$ for some w, v

Claim: $x \subset y$ is p.r.

Pf: $x \subset y$ iff $y = wxv$
iff $\underbrace{\exists u|y\ (x^* \mid u^*)}_{\text{is pr.}}$
$x^* = \text{Rev}(x)$, $u^* = \text{Rev}(u)$
think $u = wx$

④ if $P(y, \vec{x})$ is pr and we define
$Q(z, \vec{x})$ iff $\exists y \subset z\ P(y, \vec{x})$
then Q is p.r.

Pf. Observe $Q(z, \vec{x})$ iff
$\exists w|z\ (y^* | w^* \wedge P(y, \vec{x}))$
pr, pr, pr ✓

5 ("bdd maximalization")

If $P(y, \vec{x})$ is p.r. and we define:

$f(y, \vec{x}) := \mu z|_y \, P(z, \vec{x})$

write $:=$ longest init sgmt of y s.t. $P(z, \vec{x})$ (if exists)

$= 0$ o/w

then f is p.r.

Pf: you try. ✓

- now we do some ex's in specific case when $A^* = \mathbb{N}$
- i.e. sps $A = \{\sigma_1\} = \{\text{"}1\text{"}\}$

$A^* = \{0, 1, 11, 111, \ldots\}$

$= \{0, 1, 2, 3 \ldots\}$

Prop'n: The following are p.r.

1. $n + m$

Pf: $0 + m = 1$

$(n+1) + m = (n+m)+1$

i.e. $f(0, x) = x \quad f(y\sigma_1, x) = S_1(P^3_1(f(y,x), y, x))$

2. $n \cdot m$

Pf: $0 \cdot m = 0$

$(n+1) \cdot m = n \cdot m + m$ ← pr by 1 different use of exponents than before

(3) m^n

<u>Pf</u>: $m^0 = 1$

$m^{n+1} = m^n \cdot m$ ← pr by (2)

(4) $m = n$

<u>Pf</u>: above

(5) $m \leq n$

<u>Pf</u>: above (is same as $m | n$)

(6) $m | n$ i.e. "m divides n"

(~~notation~~ bad notation alert!!)

$R(m, n)$ iff m divides n

<u>Pf</u>: $m | n$ iff

$\exists k \leq n \; (n = m \cdot k)$

p.r. p.r p.r

(7) Define $n/m := \frac{n}{m}$ when $m | n$

$= 0$ o/w

then n/m is p.r.

<u>Pf</u>: - define $R(n, m, k)$ iff $n = m \cdot k$

then R is p.r. rel'n

- define

$f(n,m) = \mu k \leq n \; R(n,m,k)$ if $\exists k \leq n \; R(n,m,k)$

$= 0$ o/w

p.r.

then f is (piecewise) pr and $= n/m$ as defined above.

(8) $n \dot{-} m := \begin{cases} n - m & \text{iff } n \geq m \\ 0 & \text{o/w} \end{cases}$

is p.r.

Pf: last time

(9) $n!$

Pf: $0! = 0$

$(n+1)! = (n+1)\, n!$

(10) $\mathrm{Min}_k(n_1, \dots, n_k) := \min \text{ of } \{n_1, \dots, n_k\}$

is p.r.

Pf: by induction on k once we have Min_2

$\mathrm{Min}_2(n, m) := \begin{cases} n & \text{if } n \leq m \\ m & \text{o/w} \end{cases}$

p.r. ✓

$\mathrm{Max}_k(n_1, \dots, n_k)$ also p.r.

(11) $sg(n) := \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{o/w} \end{cases}$

$sg'(n) = 1 - sg(n)$ (there are Lev, Lev′)

(12) $|n - m|$

Pf: you try

(13) $Rm(m,n) :=$ remainder r of m when divided by n

$:= m \pmod{n}$ when $n > 0$

$= 0$ $n = 0$

Pf: $Rm(m,n) = \mu r \le n-1 \; (\exists q \le m \, (m = nq + r))$

everything p.r.

(14) $Qt(m,n) :=$ quotient q of m when divided by n

Pf: you try.