Ex: if $g(y,\vec{x}) \equiv \sigma_1$ (constant σ_1 function) then $f = \emptyset$ (since g never 0)
$\leftarrow$ empty (partial) function

Def'n A partial function $f: (A^*)^n \to A^*$ is recursive if it can be obtained from initial functions using composition, primitive recursion, and minimalization

e.g. the empty function $\emptyset$ is recursive, as above ex. shows, but not primitive recursive, since all p.r. functions are total.

Def'n an n-ary relation $R \subseteq (A^*)^n$ is recursive iff χ_R is recursive

assumed to be total

$\chi_R(\vec{x}) = 0$ means $\vec{x} \in R$
$\chi_R(\vec{x}) = 1$ $x \notin R$

Prop'n ① all p.r. functions are recursive
② Recursive rel'ns are closed under $\neg, \wedge, \vee$ and substitution by total recursive functions (i.e. if $P(x_1, \ldots, x_n)$ recursive and $f_1(\vec{y}), \ldots, f_n(\vec{y})$ total recursive then $P(f_1(\vec{y}), \ldots, f_n(\vec{y}))$ recursive)

(5) Recursive functions are closed under piecewise def'n with recursive clauses
(4) " " are closed under minimalization (by def'n)
(8) over $\mathbb{N}$, recursive functions are closed under simultaneous recursion
(6) a partial $f: (A^*)^n \to A^*$ is recursive iff $\hat{f}: \mathbb{N}^n \to \mathbb{N}$ is recursive
(7) Given $A \subseteq B$, $f: (A^*)^n \to A^*$ is recursive iff $\exists$ recursive $g: (B^*)^n \to B^*$ s.t. $g \upharpoonright A^* = f$.

pf: omitted. (each either follows from def'n or by similar args for analogous facts about pr functions)

The Ackermann function

↳ $\exists$ recursive functions that are not pr: any non-total recursive function (e.g. $\emptyset$)

↳ Q: are there total functions that are recursive but not pr?

↳ yes! we introduce one - the "Ackermann function" - which in a sense grows faster than any pr function.

Note: in what follows, working w/ functions on $\mathbb{N}$ ($= \{\sigma_1\}^*$)

- for $f: \mathbb{N} \to \mathbb{N}$, write $f^{(n)}$ for nth composed iterate of f.

$$f^{(0)}(x) := x$$
$$f^{(1)}(x) := f(x)$$
$$f^{(2)}(x) := f(f(x))$$
$$f^{(3)}(x) := f(f(f(x))) \quad \text{etc...}$$

- for f p.r. the function $F(n,x) := f^{(n)}(x)$ is p.r:

def'n by p.r.
$$\begin{cases} F(0,x) := x \\ F(n+1,x) := f(F(n,x)) \end{cases}$$

- we will define a sequence of p.r. functions $A_n: \mathbb{N} \to \mathbb{N}$ s.t. the function $A: \mathbb{N}^2 \to \mathbb{N}$

$$A(n,x) = A_n(x) \quad \text{is \underline{not} pr}$$

(but recursive!)

- define $A_0: \mathbb{N} \to \mathbb{N}$ by $A_0(x) = x+1$
- given $A_n: \mathbb{N} \to \mathbb{N}$ define $A_{n+1}(x) := A_n^{(x+1)}(1)$

↳ so roughly: input x tells us # of times to iterate prev. function to get output.

- We have:

$$A_0(x) = x+1$$

$$A_1(x) = A_0^{(x+1)}(1)$$

$$= 1 + \underbrace{(1+1+ \cdots +1)}_{x+1 \text{ times}}$$

$$= x+2$$

$$A_2(x) = A_1^{(x+1)}(1)$$

$$= 1 + \underbrace{(2+2+ \cdots +2)}_{x+1} = 2x+3$$

$$A_3(x) = A_2^{(x+1)}(1)$$

$$2(\cdots(2(2(1)+3)+3)\cdots)+3$$

exponential...

$$A_4(x) = exp^{exp^{\cdot^{\cdot^{\cdot}}}}$$

$\vdots$

- We claim each A_n is pr;
- true for $n=0$ since $x+1$ is pr.
- Sps true for n.
- can define A_{n+1} by prim recurs'n:

$$A_{n+1}(0) = A_n^{(1)}(1) = A_n(1) \quad (\text{constant})$$

$$A_{n+1}(x+1) := A_n^{(x+2)}(1)$$

$$= A_n(A_n^{(x+1)}(1))$$

$$= A_n(A_{n+1}(x))$$

... which is of form $f(A_{n+1}(x), x)$ for f pr, since A_n pr.

remember this identity

we'll use

by induction: A_n pr $\forall n \in \mathbb{N}$

- following theorem says roughly that every pr function can be bounded by some A_n

<u>Theorem</u> if $F(x_1, .., x_k)$ is pr (on $\mathbb{N}$) then there is M s.t. for all $\vec{x} \in \mathbb{N}^k$ we have $F(\vec{x}) < A_M(\max(\vec{x}))$

<u>Pf</u>: - by <u>induction</u> <u>on</u> <u>construction</u> of F

(Initial functions)

- if $F(x) = 0$ (i.e. F is erase) then $F(x) < A_0(x) = x+1$ $\forall x \in \mathbb{N}$ (so $M=0$ works)
- if $F(x) = x+1$ (i.e. F is successor on $\mathbb{N}$) then $F(x) < A_1(x) = x+2$ $\forall x \in \mathbb{N}$ (so $M = 1$ works)
- if $F(x_1, .., x_k) = x_i$ (projection @ i) then $F(\vec{x}) < A_0(\max(\vec{x})) = \max(\vec{x}) + 1$ (so $M=0$ works)

(Inductive step: composition and primitive recursion)

↳ deferred for now. ✓

- finishing pf of thm requires some work; let's just assume it for now.
- Can use theorem to show $A(n,m)$ is not p.r.

Lemma $\forall n \forall x$ $A_n(x)$ is strictly increasing in x, and $A_n(x) > x$

Pf: we induct on n
- if $n = 0$, $A_0 = x+1$ — both statements clear
- Sps true for n consider A_{n+1}

$$A_{n+1}(x+1) = A_n^{(x+2)}(1)$$
$$= A_n(A_n^{(x+1)}(1))$$
$$> A_n^{(x+1)}(1) \quad \text{by IH: } A_n(x) > x \ \forall x$$
$$= A_{n+1}(x)$$

$\Rightarrow A_{n+1}(0) < A_{n+1}(1) < \ldots$ i.e. $A_{n+1}(x)$ incr. in x

- but $A_{n+1}(0) = A_n(1) > 1 > 0$
- and since by above A_{n+1} is increasing in x (and has values in $\mathbb{N}$) that

$A_{n+1}(1) > 1 \qquad A_{n+1}(2) > A_{n+1}(1) \geq 2$

$A_{n+1}(3) > A_{n+1}(2) \geq 3$ etc.

so $A_{n+1}(x) > x \quad \forall x$

Corollary the function $A: \mathbb{N}^2 \to \mathbb{N}$ defined by $A(n,x) = A_n(x)$ is <u>not</u> p.r.

Pf - Sps A is pr
- then $A' : \mathbb{N} \to \mathbb{N}$ defined by $A'(x) = A(x,x)$
$= A_x(x)$

is p.r. (composition of pr functions)
- by Theorem, $\exists M$ s.t. $A'(x) < A_M(x)$ for every $x \in \mathbb{N}$.
- but then $A_M(M) = A'(M) < A_M(M)$ a contradiction. ✓

↳ remains to prove inductive step in Theorem
- need a few lemmas

Lemma 1 $\quad A_n(x+1) \leq A_{n+1}(x)$

Pf: fix n, induct on x.
- for $x = 0$, $\quad A_n(1) = A_{n+1}(0)$ ✓ (def'n)
- sps for x
- for $x+1$:

$$A_{n+1}(x+1) = A_n(A_{n+1}(x)) \quad \text{(as computed before)}$$
$$\geq A_n(A_n(x+1)) \quad \text{by IH + prev. lem}$$

and since $A_n(x+1) > x+1$ (also proved)

hence $\geq x+2$

get

$$\geq A_n(x+2) = A_n(x+1)+1$$

what we wanted to show ✓

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- notice that Lemma 1, when combined w/ prev lem gives $A_n(x)$ is strictly increasing in n since:

$$A_n(x) \overset{\text{prev lem}}{<} A_n(x+1) \overset{\text{lem 1}}{\leq} A_{n+1}(x)$$

Lemma 2 $\forall n \forall y$: $A_n(2y) < A_{n+2}(y)$

Pf: fix n, induct on y

- $y=0$:

$$A_{n+2}(0) = A_{n+1}(1) = A_n^{(2)}(1)$$
$$= A_n(A_n(1))$$
$$> A_n(A_n(0)) \quad \text{(prev lem)}$$
$$> A_n(0) = A_n(2\cdot 0) \quad \text{(prev lem)}$$

- Sps $A_n(2y) < A_{n+2}(y)$
- $y+1$:

$$A_{n+2}(y+1) = A_{n+1}(A_{n+2}(y))$$
$$> A_{n+1}(A_n(2y)) \quad \text{IH}$$
$$\geq A_n(A_n(2y)+1) \quad \text{Lemma 1}$$
$$\geq A_n(2y+2)$$

since $A_n(2y) \geq 2y+1$

$$= A_n(2(y+1)), \text{ as desired}$$

↳ now we can finish pf of Theorem.

↳ remains to do inductive steps (composition, primitive recursion)

Pf (continued):

(i) Composition: Spr $F(\vec{x}) = g(h_1(\vec{x}), \ldots, h_k(\vec{x}))$