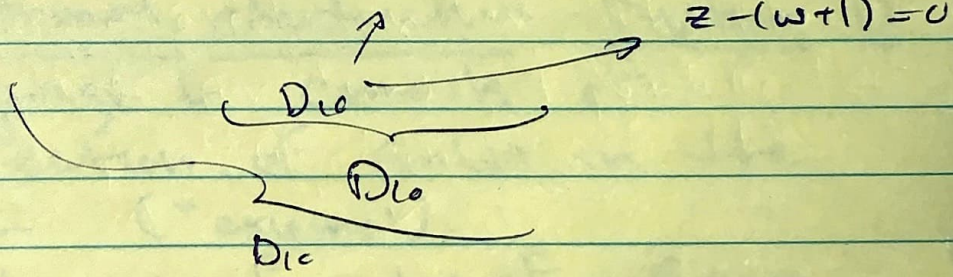
## The non "$\forall x<y$" parts:

- $\exists$: clear
- $\wedge$ and $\vee$: if $R(\vec{x})$, $S(\vec{x})$ defined by $\exists \vec{z}(F(\vec{x},\vec{z})=0)$ and $\exists \vec{z}\,(G(\vec{x},\vec{z})=0)$ resp. (w/ $F, G$ poly's)
  then: $\exists \vec{y}\,\exists \vec{z}\,(F^2(\vec{x},\vec{y})+G^2(\vec{x},\vec{z})=0)$ defines $R(\vec{x}) \wedge S(\vec{x})$
  and: $\exists \vec{y}\,\exists \vec{z}\,(F(\vec{x},\vec{y})\cdot G(\vec{x},\vec{z})=0)$ defines $R(\vec{x}) \vee S(\vec{x})$

- substitution: if $R(\vec{x},y)$ and $f(\vec{z})$ are $D_0$ then $(\vec{x},\vec{z}) \in R(\vec{x}, f(\vec{z}))$ iff $\exists y\,(R(\vec{x},y) \wedge f(\vec{z})=y)$ which is $D_0$ by above

  ← remember: we're quantifying over $\mathbb{N}$.

- $x<y$ iff $\exists z \exists w\,(y-(x+z)=0 \wedge$ [illegible] $)$




- to show $\beta$ is $D_0$ read:

  **Lemma**: $x \equiv y \pmod z$ is $D_0$

  Pf. $x \equiv y \pmod z$ iff $\exists m((x-y)-mz=0 \vee (y-x)-mz=0)$ ✓

  **Corollary**: $\beta(a,b,c)=k$ is $D_0$

  Pf: iff $(a \equiv k \bmod b(c+1)+1) \wedge (k < b(c+1)+1)$ ✓

- so remains to show closure under $\forall x <_y \ldots$

↳ we'll skip the proof: would take 4+ lectures

↳ a key step is showing that the exponential function $(x,y) \mapsto x^y$ is Diophantine ("Matiyasevich's lemma")

↳ if interested, I've posted my handwritten proof from '22 on Canvas page.

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... onward toward Gödel's thms. --

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## Deductive Systems and the Completeness Theorem

**Def'n** A formal deductive system $\mathcal{D}$ for a language $L$ consists of:

① a collection of fmlas in the language $L$ ("axioms")

② a finite set of rules of inference: i.e. a finite set of relations $D_i(\ell_1, \ldots, \ell_{n_i}, \chi)$ on the fmlas in $L$

↳ $D_i(\ell_1, \ldots, \ell_{n_i}, \chi)$ intuitively means $\chi$ can be deduced from the fmlas $\ell_1, \ldots, \ell_{n_i}$ by applying $D_i$

↳ We say $\psi$ is a direct consequence of $\varphi_1, \dots, \varphi_{n_i}$ if $D_i(\varphi_1, \dots, \varphi_{n_i}, \psi)$ for some $i$.

↳ a formal proof is a finite sequence of formulas $\varphi_1, \dots, \varphi_N$ s.t. each formula is either an axiom or a direct consequence of some finite preceding it.

(in this formal system)

↳↳ $\varphi$ is a formal theorem if there is a formal proof $\varphi_1, \dots, \varphi_N$ with $\varphi = \varphi_N$.

- we write $\vdash_D \varphi$, or just $\vdash \varphi$, if there is such a proof.

- If $\Sigma$ is a collection of $L$-formulas then a formal proof from $\Sigma$ is the same as before, except now we're allowed to use finite formulas from $\Sigma$ (as well as axioms) in our proof.
- If $\varphi$ can be formally proved from $\Sigma$ we write $\Sigma \vdash \varphi$.

Facts: ① If $\Sigma \subseteq \Sigma'$ and $\Sigma \vdash \varphi$ then $\Sigma' \vdash \varphi$

② If $\Sigma \vdash \varphi$ then for some finite $\Sigma_0 \subseteq \Sigma$ we have $\Sigma_0 \vdash \varphi$

③ If $\Sigma' \vdash \varphi$, and for all $\psi \in \Sigma'$ we have $\Sigma \vdash \psi$, then $\Sigma \vdash \varphi$

Pf: ①, ③ are clear; ② holds since proofs are of finite length.

# A formal deductive system for first-order logic

- We describe a deductive system w/ just a single deduction rule (modus ponens)
- Works for any language

- Fix a language $L$
- if $\varphi$ is an $L$-fmla, a generalization of $\varphi$ is any fmla of the form $\forall x_1 \ldots \forall x_k \varphi$

Our axioms are all generalizations of all fmlas of the following forms.

① Propositional axioms:

(a.) $\varphi \Rightarrow (\psi \Rightarrow \varphi)$

(b.) $(\varphi \Rightarrow (\psi \Rightarrow \eta)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \eta))$

(c.) $(\neg\varphi \Rightarrow \neg\psi) \Rightarrow ((\neg\varphi \Rightarrow \psi) \Rightarrow \varphi)$

② Quantifier axioms:

(a.) $\forall x(\varphi(x) \Rightarrow \psi(x)) \Rightarrow ((\forall x \varphi(x)) \Rightarrow (\forall x \psi(x)))$

(b.) $\varphi \Rightarrow \forall x \varphi$, if $x$ not free in $\varphi$

(c.) $\forall x \varphi(x) \Rightarrow \varphi(x|t)$

← fmla in which all free instances of $x$ replaced w/ $t$.

where $t$ is a term substitutable for $x$

i.e. no free occurrence of $x$ is in scope of $\forall z$, for some $z$ appearing in $t$.

(ex: if $t$ is $x+y$ then
$\forall x(x=x) \Rightarrow (x+y = x+y)$
is an axiom, whereas:
$\forall x \exists y\,(x = y+1) \Rightarrow \exists y(x+y = y+1)$
is not - $t$ is not subst'able for $x$ in (abstm)

## (3) Equality axioms

(a.) $x = x$
$x = y \Rightarrow y = x$
$x = y \wedge y = z \Rightarrow x = z$

(b.) For every $n$-ary rel'n symbol $R$ in $L$:
$(x_1 = y_1 \wedge \cdots \wedge x_n = y_n) \Rightarrow$
$[R(x_1, \ldots, x_n) \Rightarrow R(y_1, \ldots, y_n)]$

(c.) For every $n$-ary function symbol $F$.
$(x_1 = y_1 \wedge \cdots \wedge x_n = y_n)$
$\Rightarrow [F(x_1, \ldots, x_n) = F(y_1, \ldots, y_n)]$

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- Most of these intuitive axioms
- Turn out to be enough to get by:
  Why? Have to try, to see
- missing some axioms you might expect: e.g. $\varphi \Rightarrow \varphi$ or $\varphi \Rightarrow \neg\neg\varphi$
- but these can be proved

- Not hard to check: all axioms are tautologies: i.e. if $\tau$ is an axiom and $\mathcal{A}$ is any $L$-structure then $\mathcal{A} \models \tau$.

**Def'n** We use only one deduction rule: $D_1(\varphi, \varphi \Rightarrow \psi, \psi)$
i.e. "From $\varphi$ and $\varphi \Rightarrow \psi$, deduce $\psi$"

↳ will usually write MP instead of $D_1$
↳ using a single deduction is nice but makes some proofs awkward since we don't have direct deductions for fmlas written using $\wedge$'s, $\vee$'s, etc.

↳ **A key point**: axioms, and formal pfs, are syntactic things.

**Ex**: Sps $\Sigma \vdash \psi$ and $\Sigma \vdash \neg\psi$ for some $\psi$. Then $\Sigma$ proves every fmla.

**Pf**: - Fix a formula $\varphi$
- the following is a formal proof of $\varphi$ from $\Sigma$:

① (1.a) $\psi \Rightarrow (\neg\varphi \Rightarrow \psi)$
② (1.a) $\neg\psi \Rightarrow (\neg\varphi \Rightarrow \neg\psi)$
③ $\Sigma : \psi$
④ $\Sigma : \neg\psi$

} put pfs for $\psi$ and $\neg\psi$ here

⑤ MP(1,3): $\neg\varphi \Rightarrow \chi$
⑥ MP(2,4): $\neg\varphi \Rightarrow \neg\chi$
⑦ (1.c) $(\neg\varphi \Rightarrow \neg\chi) \Rightarrow ((\neg\varphi \Rightarrow \chi) \Rightarrow \varphi)$
⑧ MP(6,7): $(\neg\varphi \Rightarrow \chi) \Rightarrow \varphi$
⑨ MP(5,8): $\varphi$ ✓

More examples: (i) $\vdash \varphi \Rightarrow \varphi$
(ii) $\vdash \neg\neg\varphi \Rightarrow \varphi$

Pf: (i) ①(1.a) $\varphi \Rightarrow ((\varphi \Rightarrow \varphi) \Rightarrow \varphi)$ ("$\chi$")
② (1.b) $(\varphi \Rightarrow ((\varphi \Rightarrow \varphi) \Rightarrow \varphi)) \Rightarrow ((\varphi \Rightarrow (\varphi \Rightarrow \varphi)) \Rightarrow (\varphi \Rightarrow \varphi))$ ("$\chi$", "$\psi$")
③ MP(1,2): $(\varphi \Rightarrow (\varphi \Rightarrow \varphi)) \Rightarrow (\varphi \Rightarrow \varphi)$
④ (1.a) $\varphi \Rightarrow (\varphi \Rightarrow \varphi)$ ("$\chi$")
⑤ MP(3,4) $\varphi \Rightarrow \varphi$ ✓

(ii): you try ✓

Lemma (Deduction lemma):
If $\Sigma \cup \{\varphi\} \vdash \chi$ then $\Sigma \vdash \varphi \Rightarrow \chi$

Pf: - let $\varphi_1, \ldots, \varphi_n$ be a deduction (i.e. formal pf) from $\Sigma \cup \{\varphi\}$
- we check that $\forall i \le n$ we have $\Sigma \vdash \varphi \Rightarrow \varphi_i$
- so in particular if this is a deduction of $\varphi_n = \chi$, we're done.