- if not, let i be least s.t. $\Sigma \nvdash \varphi \Rightarrow \varphi_i$
- Three possibilities:

① φ_i is an axiom in Σ

Then by (1.a) have $\varphi_i \Rightarrow (\varphi \Rightarrow \varphi_i)$ is an axiom so by MP get $\varphi \Rightarrow \varphi_i$ (i.e. $\Sigma \vdash \varphi \Rightarrow \varphi_i$)

② φ_i is φ. Then by example ⓘ above get $\vdash \varphi \Rightarrow \varphi_i$, and hence $\Sigma \vdash \varphi \Rightarrow \varphi_i$

③ φ_i is deduced by applying MP to fmlas φ_k, $\varphi_k \Rightarrow \varphi_i$ appearing previously in the deduction

Then by minimality of i we get $\Sigma \vdash \varphi \Rightarrow \varphi_k$ and $\Sigma \vdash \varphi \Rightarrow (\varphi_k \Rightarrow \varphi_i)$

By (1.b) we get

$$(\varphi \Rightarrow (\varphi_k \Rightarrow \varphi_i)) \Rightarrow ((\varphi \Rightarrow \varphi_k) \Rightarrow (\varphi \Rightarrow \varphi_i))$$

So by MP:

$$(\varphi \Rightarrow \varphi_k) \Rightarrow (\varphi \Rightarrow \varphi_i)$$

So by MP again

$(\varphi \Rightarrow \varphi_i)$, contradiction ✓

<u>Theorem</u> (Gödel's completeness theorem) <u>(first form)</u>. For any Σ and φ we have

$$\Sigma \models \varphi \text{ iff } \Sigma \vdash \varphi$$

(Recall: $\Sigma \models \varphi$ means: for every L-structure $\mathcal{A}$, if $\mathcal{A} \models \Sigma$ then $\mathcal{A} \models \varphi$. Also: if φ has free variables $x_1, \dots, x_n$ then $\mathcal{A} \models \varphi$ means $\mathcal{A} \models \forall x_1 \dots \forall x_n \varphi$)

($\Leftarrow$) is straightforward direction — "soundness" — if Σ proves φ then φ is true in every model of Σ

($\Rightarrow$) is harder — "completeness" — if φ is true in every model of Σ, then can actually be proved from Σ!

<u>Pf of soundness</u>: Sps $\Sigma \vdash \varphi$ and $A \models \Sigma$

WTS $A \models \varphi$.

<u>Key point</u>: in general, if $A \models \tau(\vec{a})$ for every tuple $\vec{a}$, and also $A \models \tau(\vec{a}) \Rightarrow \sigma(\vec{a})$ $\forall \vec{a}$, then $A \models \sigma(\vec{a})$ $\forall \vec{a}$.

Pf: clear.

if φ_i is $\varphi_i(\vec{x})$ has free variables then $A \models \varphi_i$ means $A \models \varphi_i(\vec{a})$ for all $\vec{a}$

- Sps $\varphi_1, \ldots, \varphi_n$ is a deduction of $\varphi = \varphi_n$ from Σ. We prove $A \models \varphi_i$ $\forall i \leq n$
- if not, let i be least s.t. $A \not\models \varphi_i$
  - if φ_i is an axiom, then it is a tautology. Hence $A \models \varphi_i$, contradiction
  - if $\varphi_i \in \Sigma$ then $A \models \varphi_i$ by hypothesis, contradiction.
  - if φ_i is deduced by MP from φ_k and $\varphi_k \Rightarrow \varphi_i$ then by minimality of i we have $A \models \varphi_k$ and $A \models \varphi_k \Rightarrow \varphi_i$
  - Hence $A \models \varphi_i$ by "key point" contradiction ✓

- Before considering ($\Rightarrow$) (i.e. the "completeness" direction) let's see some consequences.

<u>Corollary</u> (Compactness theorem)
If $\Sigma \models \varphi$ then there is a finite $\Sigma_0 \subseteq \Sigma$ s.t. $\Sigma_0 \models \varphi$

<u>Pf</u>: By completeness theorem we know $\Sigma \vdash \varphi$.

By finiteness of proofs we know there is a finite $\Sigma_0 \subseteq \Sigma$ s.t. $\Sigma_0 \vdash \varphi$.

Then $\Sigma_0 \models \varphi$ by soundness. ✓

- We can reformulate the completeness theorem as follows:

<u>Def'n</u> Σ is <u>consistent</u> if there is no sentence φ s.t. $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg\varphi$.

- By our prev. example, Σ is inconsistent iff Σ proves <u>every</u> fmla
- <u>Observe</u>: if Σ is inconsistent, then Σ has no models (... can't have $A \models \varphi$ and $A \models \neg\varphi$ by def'n of $\models$)
  - ↪ converse also true!

<u>Thm</u> (Completeness theorem, 2nd form)
Σ is consistent iff Σ has a model

<u>Prop'n</u> The two forms of the completeness theorem are equivalent (i.e. the hard directions are equivalent, i.e. ① "$\Sigma \models \varphi \Rightarrow \Sigma \vdash \varphi$" equiv to ② "$\Sigma$ consistent $\Rightarrow \Sigma$ has a model")

<u>Pf</u>: - Sps first form (i.e. ①), and sps Σ is consistent
- if it had no model then $\Sigma \models \varphi$ and $\Sigma \models \neg\varphi$ for any φ (trivially)
- hence by ① we have $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg\varphi$ (again, for any φ), contradicting consistency.

- Now sps 2nd form (i.e ②) and sps $\Sigma \models \varphi$
- then $\Sigma \cup \{\neg\varphi\}$ has no model (why?)
- hence $\Sigma \cup \{\neg\varphi\}$ inconsistent by ②
- hence $\Sigma \cup \{\neg\varphi\} \vdash \psi$ and $\Sigma \cup \{\neg\varphi\} \vdash \neg\psi$ for some (equiv any) ψ.
- but then by deduction lemma, $\Sigma \vdash \neg\varphi \Rightarrow \psi$ and $\Sigma \vdash \neg\varphi \Rightarrow \neg\psi$
- but then by axiom 1.c. and MP (applied twice) we get $\Sigma \vdash \varphi$ ✓

- reformulation of completeness in this way led to a more elegant pf than Gödel's original
- goes by showing: if Σ consistent then can build a model for Σ called a "Henkin model" ... we omit proof ✓