we'll need:

Lemma For every $m, n, p \in \mathbb{N}$ we have:

1. $Q_0 \vdash \forall x\, (x < \underline{n+1} \Rightarrow (x = \underline{0} \vee x = \underline{1} \vee \dots \vee x = \underline{n}))$
2. if $m+n=p$ then $Q_0 \vdash \underline{m} + \underline{n} = \underline{p}$
3. if $m \cdot n = p$ then $Q_0 \vdash \underline{m} \cdot \underline{n} = \underline{p}$
4. if $m \neq n$ then $Q_0 \vdash \neg(\underline{m} = \underline{n})$

- we'll provide informal proofs from the axioms Q_0 and assume our deduction system is strong enough to turn these into formal pfs (it is)
- think like this: we can reason "semantically" as usual as though we're in some number system satisfying Q_0
  - ↳ might not be the usual one $\mathbb{N}$, but as long as we can tie any intuition about $\mathbb{N}$ back to the axioms in Q_0 our proofs will be proofs from Q_0.

Pf of Lemma: ① induction on n

For $n = 0$: follows from Ax's ⑤ + ⑥ (substituting 0 for y in ⑤)

For $n+1$: by Ax⑤: $Q_0 \vdash \forall x\, (x < \underline{n+2} \Rightarrow x < \underline{n+1} \vee x = \underline{n+1})$ which by induction gives $Q_0 \vdash \forall x\, (x < \underline{n+2} \Rightarrow x = \underline{0} \vee \dots \vee x = \underline{n} \vee x = \underline{n+1})$ ✓

2. induction on n using Ax 7
3. induction on n using Ax 8
4. induction on m

If $m=0$, then $n \neq 0 \Rightarrow n = S(k)$ for some k by Ax. 2. So $Q_0 \vdash \underline{m} \neq \underline{n}$ by Ax. 1

i.e. $S(m)$ For $m+1$, if $n=0$ then done by same arg. If $n \neq 0$ then $n = S(k) = k+1$ for some k (Ax. 2), so $m+1 \neq k+1$ (i.e. $S(m) \neq S(k)$) so $m \neq k$ by Ax. 3. Then by induction $Q_0 \vdash \underline{m} \neq \underline{k}$

so $Q_0 \vdash \underline{m+1} \neq \underline{k+1}$ by Ax 3 ✓

Pf of theorem: We show every min-recursive f is rep'able

By HW3, min-rec $\Leftrightarrow$ rec, so this suffices

Recall: $f: \mathbb{N}^k \to \mathbb{N}$ is min-rec iff built up from the basic functions:

$x \mapsto 0$

$+$

$\cdot$

$(x_1, \ldots, x_k) \mapsto x_i$ $(\forall i \leq k)$

$\chi_=$ (char. function for $=$)

using composition and minimalization.

Also recall: to check "$f(x)=y$" is rep'able by a fmla φ, WTS $Q_0 \vdash \forall x,y,z\,(\varphi(x,y) \wedge \varphi(x,z) \Rightarrow y=z)$ and $Q_0 \vdash \varphi(\underline{n},\underline{m})$ whenever $f(\underline{n}) = \underline{m}$

① Basic functions are Q_0-rep'able:

(i) $x \mapsto 0$ rep'd by $\varphi(x,y) := y=0$

why: $Q_0 \vdash \forall x,y,z\,(y=0 \wedge z=0 \Rightarrow y=z)$ follows from axioms for $=$ in our deduction system (don't need any Q_0 axioms)

Also, $Q_0 \vdash \varphi(\underline{x},\underline{0})$ for all $x \in \mathbb{N}$ since $\varphi(\underline{x},\underline{0})$ is just $0=0$ (since $\underline{0}$ is 0) which again follows from $=$ axioms

(ii) $+$ is rep'able by $\varphi(x_1,x_2,y) := x_1+x_2=y$

why: Unique output property can be proved using transitivity axioms for $=$. And if $n+m=k$ then $Q_0 \vdash \underline{n}+\underline{m}=\underline{k}$ by lemma.

(iii) $\cdot$ is rep'able by $\varphi(x_1,x_2,y) := x_1 \cdot x_2 = y$

pf similar to (ii)

(iv.) $(x_1, \ldots, x_k) \mapsto x_i$ rep'able by $\varphi(x_1, \ldots, x_k, y) := y = x_i$. Axioms for $=$ show φ works.

(v.) $\chi_=$ rep'able by $\varphi(x_1,x_2,y) := (x_1 = x_2 \wedge y = \underline{1}) \vee (x_1 \neq x_2 \wedge y = \underline{0})$

why: unique output prop follows from $=$ axioms.

And if $\chi_=(x_1,x_2)=e$ then $Q_0 \vdash$
$\varphi(\underline{x_1},\underline{x_2},\underline{e})$ by part ④ of Lemma
(try to spell this out yourself!) ✓

② Now must show Q_0-rep'ble functions are closed under comp'n and min'z'n.

<u>Composition</u>: Sps g and $h_1,\dots,h_k$ are rep'ble by ψ and $\varphi_1,\dots,\varphi_k$
and $f(x_1,\dots,x_n)=g(h_1(\vec{x}),\dots,h_k(\vec{x}))$
Then f is rep'ble by
$\tau(\vec{x},y):=\exists z_1\dots z_k(\varphi_1(\vec{x},z_1)\wedge\dots\wedge\varphi_k(\vec{x},z_k)\wedge\psi(z_1,\dots,z_k,y))$

<u>Why</u>: uniq output prop follows from uniq. out. prop of the φ_i and ψ.
and if $f(a_1,\dots,a_n)=m$ we know
$Q_0\vdash\varphi_i(\vec{\underline{a}})=\underline{m_i}$ and $Q_0\vdash\psi(\underline{m_1},\dots,\underline{m_k},\underline{m})$
for some m_i's, so these m_i's witness the $\exists z_i$'s and $Q_0\vdash\tau(\underline{a_1},\dots,\underline{a_n},\underline{m})$ ✓

<u>Minimalization</u>: Sps $g:\mathbb{N}^{k+1}\to\mathbb{N}$ is rep'ble by ψ and $f(x_1,\dots,x_n)=$
$\mu y[g(x_1,\dots,x_n,y)=0]$
Then f is rep'ble by:
$\varphi(\vec{x},y):=\psi(\vec{x},y,0)\wedge\underbrace{\forall w(w<y}_{\text{"}\forall w<y\text{"}}\Rightarrow\exists z(z\neq 0\wedge\psi(\vec{x},w,z)))$

<u>Why</u>: if $\varphi(\vec{x},y)$ and $\varphi(\vec{x},z)$ then $y\geq z$
and $z\geq y$ so we have (and $Q_0\vdash$) $y=z$

And if $f(a_1,..,a_n) = m$ then $g(a_1,..,a_n,m) = 0$
and $g(\bar{a}, i) = l_i > 0$ $\forall i < m$

WTS: $Q_0 \vdash \varphi(\underline{a_1},..,\underline{a_n},\underline{m})$
i.e. $Q_0 \vdash \psi(\underline{a_1},..,\underline{a_n},\underline{m},0)$, which is true since ψ reps g, and
$$Q_0 \vdash \forall w (w < \underline{m} \Rightarrow \exists z (z \neq 0 \wedge \psi(\bar{\underline{a}}, w, z)))$$
We know $Q_0 \vdash \underline{l_i} \neq 0$ by (a) of Lemma
and $Q_0 \vdash \forall w (w < \underline{m} \Rightarrow w = \underline{0} \vee w = \underline{1} \vee .. \vee w = \underline{m-1})$
by (c) of Lemma
and $Q_0 \vdash \psi(\bar{\underline{a}}, \underline{i}, \underline{l_i})$ for each $i < m$
again since ψ reps g
so indeed $Q_0 \vdash \forall w (w < \underline{m} \Rightarrow \exists z (z \neq 0 \wedge \psi(\bar{\underline{a}}, w, z)))$ ✓

✓

Def'n A set of sentences Q is undecidable if $\langle Conseq(Q) \rangle$ is not recursive.

Theorem (Gödel's 1st Incompleteness Thm - abstract existence form)
Sps Q is a recursive set of axioms in the language of arithmetic and $Q_0 \subseteq Q$. If Q is consistent, then Q is incomplete and undecidable.

Pf. Since Q_0 reps every recursive F/R so does Q. By theorem from last

Lecture, this gives $\langle Conseq(Q) \rangle$ is not Q-repable, hence not recursive (i.e. Conseq(Q) is not recursive).

We showed Conseq(Q) is always r.e. when Q recursive, and if Q is complete then Conseq(Q) is actually recursive. Hence Conseq(Q) is incomplete (so Q is too).

↳ this pf nicely uses our complexity theory; later we'll give another pf in which we produce an explicit σ s.t. $Q \nvdash \sigma$ and $Q \nvdash \neg\sigma$.

Briefly: Interpretations

- Gödel's thm is more powerful than it appears at first.
- There are many natural examples of theories T in a language L (possibly quite different from L_{ar}) that "interpret" Q_0.
- This means roughly that for every fmla $\varphi \in Q_0$ we can find an "L-version" φ^* s.t. $T \models \varphi^*$ (and hence $T \vdash \varphi^*$)
- if we let $Q_0^* = \{\varphi^* : \varphi \in Q_0\}$ we get $T \vdash Q_0^*$