- <u>minimalization</u>: Sps we have a unary $f(x) = \mu t[g(x,t) = 0]$ and $g(x,t) = y$ iff $\exists k\, P(x,t,y,\bar{\alpha}(k))$

for simplicity

then $f(x) = y$ iff $g(x,y) = 0 \wedge \forall z<y\ g(x,z) > 0$ (and defined)

iff $\exists k(P(x,y,0,\bar{\alpha}(k))$
$\wedge \forall z<y\ \exists w\,(w \neq 0 \wedge \exists \ell\, P(x,z,w,\bar{\alpha}(\ell))))$

iff $\exists k(P(\ldots) \wedge \exists w \exists \ell\, \forall z<y\,((w)_z \neq 0 \wedge P(x,z,(w)_z,\bar{\alpha}((\ell)_z))))$

code $\forall z<y\ \exists w \exists \ell$ as
$\exists$ seqs $w = \langle (w)_0, \ldots \rangle$ $\ell = \langle (\ell)_0, (\ell)_1 \ldots \rangle$ $\forall z<y \ldots$

iff $\exists s\,(P(x,y,0,\bar{\alpha}((s)_0)) \wedge \forall z<y\,(((s)_1)_z \neq 0 \wedge P(x,z,((s)_1)_z,(\bar{\alpha}(((s)_2)_z))))$

$\langle k,w,\ell \rangle$

iff $\exists s\, R(x,y,\bar{\alpha}(s))$

where R is p.r. using same tricks as above ✓ ✓

<u>Thm</u> (Kleene's normal form relative to α).
There is a p.r. $U: \mathbb{N} \to \mathbb{N}$ and a p.r. relation $T \subseteq \mathbb{N}^3$ s.t. for every oracle α,
every $f: \mathbb{N}^n \to \mathbb{N}$ in $R(\alpha)$ has the form
$U(\mu k[T(e, \langle \vec{x} \rangle, \bar{\alpha}(k))])$ for some $e \in \mathbb{N}$

Pf: Recall (11Za) there is a p.r. relation T' that is universal for recursively enumerable relations, i.e. for each r.e. $Q \subseteq \mathbb{N}^n$ there is a code $e \in \mathbb{N}$ s.t.

$$\underbrace{Q(x_1, \ldots, x_n)}_{\vec{x}} \text{ iff } \exists m\, T'(e, \langle \vec{x} \rangle, m)$$

Recall

↳ e codes a TM M whose outputs are the (encoded) elements of Q

↳ m codes a halting computation of M w/ output $\langle \vec{x} \rangle$

↳ above says: $\vec{x} \in Q$ iff $\exists$ a halting comp'n of M w/ output $\langle \vec{x} \rangle$ (i.e. ~~[illegible]~~

Now: Sps $f: \mathbb{N}^n \to \mathbb{N}$ is recursive in α

- by prev. thm. there is a p.r. P s.t.

$$f(\vec{x}) = y \text{ iff } \underbrace{\exists k\, P(\vec{x}, y, \bar{\alpha}(k))}_{\text{r.e. rel'n}}$$

- hence for some e we have:

$$f(\vec{x}) = y \text{ iff } \exists k \exists m\, T'(e, \langle \vec{x}, y, \bar{\alpha}(k) \rangle, m)$$

$$\text{iff } \exists s\, T'(e, \langle \vec{x}, y, \bar{\alpha}((s)_0) \rangle, (s)_1)$$

$$\langle k, m \rangle \to s$$

$$\text{iff } \exists s\, T''(e, \langle \vec{x}, y, \bar{\alpha}(s) \rangle)$$

where T'' is the p.r. modification of T' that makes the above true.

- Now we can compute $f(\vec{x})$ as follows:

$$f(\vec{x}) = \left(\mu k [T''(e, \langle \vec{x}, (k)_0, \bar{\alpha}((k)_1) \rangle)]\right)_0$$

" search for least code $k = \langle y, s \rangle$ s.t. $T''(e, \langle \vec{x}, y, \bar{\alpha}(s) \rangle)$ and return y "

- Hence if we let $U(z) = (z)_0$ (U p.r.) and let T be p.r. mod. of T'' s.t.:

$$T(e, \langle \vec{x} \rangle, \bar{\alpha}(k)) \text{ iff } T''(e, \langle \vec{x}, (k)_0, \bar{\alpha}((k)_1) \rangle)$$

we get

$$f(x) = U(\mu k [T(e, \langle \vec{x} \rangle, \bar{\alpha}(k))])$$

as desired ✓

<u>Corollary</u> (Kleene's enumeration thm for $R(\alpha)$) For every oracle $\alpha : \mathbb{N} \to \mathbb{N}$, $R(\alpha)$ has the enumeration property, i.e. there is a function

$$\varphi^\alpha : \mathbb{N}^2 \to \mathbb{N}, \quad \text{recursive in } \alpha$$

s.t. for every unary $f : \mathbb{N} \to \mathbb{N}$ recursive in α, there is $e \in \mathbb{N}$ s.t. $\varphi^\alpha(e, x) = f(x)$ for all x

(we'll write $\varphi^\alpha_e(x)$ for $\varphi^\alpha(e,x)$, so thm says $\exists e$ s.t. $\varphi^\alpha_e = f$)

<u>Pf.</u> let $\varphi^\alpha(e, x) = U(\mu k [T(e, x, \bar{\alpha}(k))])$

✓

really $\langle x \rangle$

Def'n (i) a relation $P \subseteq \mathbb{N}^n$ is recursive in α if χ_P is recursive in α

(ii) $P \subseteq \mathbb{N}^n$ is recursively enumerable in α (r.e. in α) if there is $Q \subseteq \mathbb{N}^{n+1}$ rec. in α s.t. $P(\vec{x})$ iff $\exists k\, Q(\vec{x}, k)$

→ rule of thumb: every thm about r/r.e. sets and functions from 117a has a "relativized to α" version.

→ above Kleene thms are examples.

→ another:

Thm There is a p.r. relation T s.t. for every α and every $P \subseteq \mathbb{N}$ r.e. in α, there is $e \in \mathbb{N}$ s.t.

$$P(x) \text{ iff } \exists k\, T(e, x, \bar{\alpha}(k))$$

Pf: you try

Turing degrees

↳ want to understand how much "extra" info a non-recursive oracle α gives us and which oracles give more or less.

Def'n Sps $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ are oracles. We say α is Turing reducible to β, and write $\alpha \leq_T \beta$, if $\alpha \in R(\beta)$