Claim for all oracles α, β, γ:

1. $\alpha \leq_T \beta$ iff $R(\alpha) \subseteq R(\beta)$
2. $\alpha \leq_T \alpha$
3. $\alpha \leq_T \beta \leq_T \gamma \Rightarrow \alpha \leq_T \gamma$

} $\leq_T$ is a "quasi-order"

PF: 2 and 3 follow from 1 and 1 follows immediately from def'ns ✓

Notation: write $\alpha \equiv_T \beta$ if $\alpha \leq_T \beta$ and $\beta \leq_T \alpha$ "Turing equivalent"

Observe: $\equiv_T$ is an equivalence rel'n on $\mathbb{N}^{\mathbb{N}}$ set of all oracles $\alpha : \mathbb{N} \to \mathbb{N}$

- write $[\alpha]_T$ for $\equiv_T$ equiv. class of α "Turing degree of α"
- write $\mathcal{D}$ for $\{[\alpha]_T : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ set of Turing degrees

- on $\mathcal{D}$, $\leq_T$ is a partial order (on $\mathbb{N}^{\mathbb{N}}$ $\leq_T$ is just a quasi-order: not anti-symmetric)

- we will sometimes write $\underset{\sim}{a}, \underset{\sim}{b}$, etc. to denote classes $[\alpha]_T, [\beta]_T$, etc in $\mathcal{D}$
- so e.g. $\underset{\sim}{a} = [\alpha]_T$ for any $\alpha \in \underset{\sim}{a}$.

Sequence notation

- write $\mathbb{N}^n$ for set of sequences $s = (s_0, \ldots, s_{n-1})$ of length n on $\mathbb{N}$
- $\mathbb{N}^{<\mathbb{N}} := \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ = set of all finite sequences on $\mathbb{N}$ (including empty sequence $\emptyset$)
- $\mathbb{N}^{\mathbb{N}}$ = all infinite sequences $\alpha = (a_0, a_1, \ldots)$
  = all oracles
- write $|s|$ for length of s
  $= n$ if $s \in \mathbb{N}^n$
  $= \infty$ if $s \in \mathbb{N}^{\mathbb{N}}$

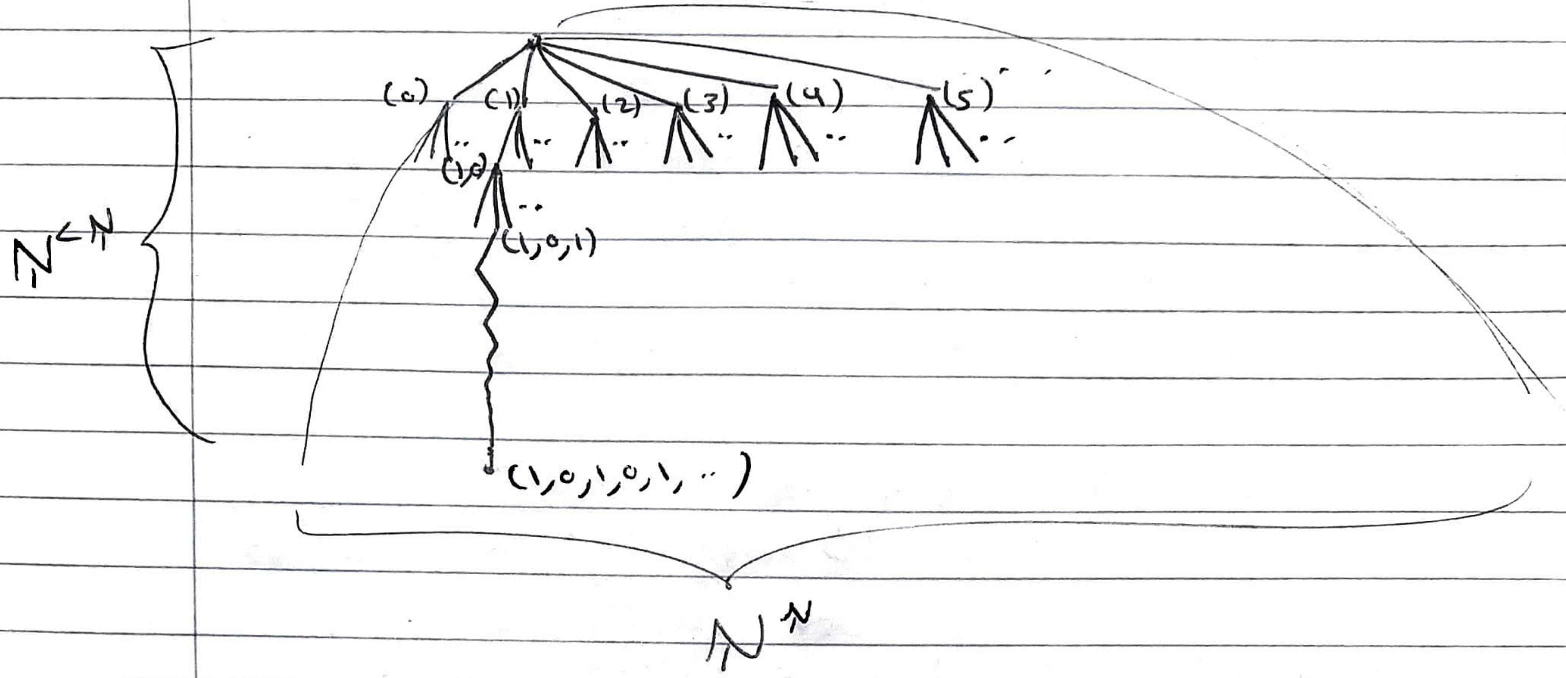
  ↳ not same as length(x) which is length of sequence coded by $x \in \mathbb{N}$
- if $|s| \geq n$ write $s \upharpoonright n$ for initial sequence of first n entries of s

  e.g. $(1,0,1,2,3) \upharpoonright 3 = (1,0,1)$
  $(1,1,1,\ldots) \upharpoonright 3 = (1,1,1)$

- if $|t| \geq |s|$ and $t \upharpoonright |s| = s$ then we say t <u>extends</u> s and write $s \leq t$
- e.g. $(1,0,1,2,3)$ extends $(1,0,1)$

- if a is a finite sequence and b is a finite or infinite sequence, write ab for a concatenated w/ b
- e.g. $(1,0)(1,2,3) = (1,0,1,2,3)$

Pic of N^N:



Another def'n of $\le_T$

- a total function $\phi : N^{<N} \to N^{<N}$ is **monotone** if $s \le t \Rightarrow \phi(s) \le \phi(t)$
- e.g. if $f : N \to N^{<n}$ is any function, can define $\phi : N^{<N} \to N^{<N}$ by

$$\phi(a_0, \dots, a_{n-1}) = f(a_0)\, f(a_1) \cdots f(a_{n-1})$$

↳ then ϕ is monotone.

↳ e.g. if $f(n) = \underbrace{(n, n, \dots, n)}_{n\text{-times}}$

then ϕ so defined is monotone and e.g.

$$\phi(2,0,3) = f(2)\, f(0)\, f(3)$$
$$= (2,2)\, \overset{\text{empty set}}{\emptyset}\, (3,3,3) = (2,2,3,3,3)$$

↳↳ not all monotone functions are of this form.

- if ϕ is monotone can define
  $\phi^*: \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by
  $$\phi^*(u) = \lim_{n\to\infty} \phi(u \restriction n) = \bigcup_{n\in\mathbb{N}} \phi(u\restriction n)$$

- def'n makes sense by monotonicity
- e.g. for ϕ above
  $$\phi^*(0,1,2,3,\ldots) = (1,2,2,3,3,3,\ldots)$$

- we consider $\phi^*(u)$ defined only for $u \in \mathbb{N}^{\mathbb{N}}$ where $|\phi(u\restriction n)| \to \infty$
  - ↳ so ϕ^* can be partial
  - ↳ e.g. $\phi^*(1,2,0,0,\ldots)$ for above is undef (or "=" $(1,2,2)$ which is $\notin \mathbb{N}^{\mathbb{N}}$)

- say that a map $\phi: \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is <u>recursive</u> if the encoded map
  $\vec{\phi}: \mathbb{N} \to \mathbb{N}$ defined by
  $\vec{\phi}(\langle s\rangle) = \langle \phi(s)\rangle$
  is recursive
  - ↳ notice: $\vec{\phi}$ is partial – only defined on codes!
  - ↳ need to work in terms of codes don't have def'n for "recursive" for functions on $\mathbb{N}^{<\mathbb{N}}$

another view of $\leq_T$

↳ <u>Theorem</u> Sps $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ are oracles. Then $\alpha \leq_T \beta$ iff there is a recursive monotone $\phi: \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ s.t. $\phi^*(\beta) = \alpha$.

Pf ($\Leftarrow$) Sps ϕ is rec. mono. w/ $\phi^*(\beta) = \alpha$

- How to compute $\alpha(n)$?
- hypothesis gives: $\forall$ large enough k s.t. $|\phi(\beta \upharpoonright k)| \geq n+1$
  $\alpha \upharpoonright n+1 = (\alpha(0), \ldots, \alpha(n))$ is $\leq \phi(\beta \upharpoonright k)$
- so: working w/ codes and $\hat{\phi}$ we have:
  $$\alpha(n) = \left(\hat{\phi}\left(\overline{\beta}\left(\mu k\left[\text{length}(\overline{\beta}(k)) > n\right]\right)\right)\right)_n$$
- shows: $\alpha \in R(\overline{\beta}) = R(\beta)$, hence $\alpha \leq_T \beta$.

($\Rightarrow$) Want: a monotone (hence total) $\phi: \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ s.t. $\phi(\text{initial seg of } \beta) = \text{init. seg. of } \alpha$

- since $\alpha \leq_T \beta$, $\alpha \in R(\beta)$, so by Kleene $\exists$ p.r. U, T and $e \in \mathbb{N}$ s.t.
  $$\alpha(n) = U(\mu k[T(e, n, \overline{\beta}(k))])$$

remember k doing a lot of work here; codes pair $((k)_0, (k)_1)$ where $(k)_0$ is $\alpha(n)$ and $(k)_1$ is length of init seg of β needed to compute $\alpha(n)$

↳ so to compute $\alpha(n)$ need to know long enough init seg of β, i.e. need k large enough s.t. $T(e, n, \overline{\beta}(k))$ is true.
↳ hence for long enough init seg of β can compute all of $\alpha(0), \ldots, \alpha(n)$
↳ but we also need to define ϕ for $s \in \mathbb{N}^{<\mathbb{N}}$ that are not init. segs of β

first attempt:

means: output is seq of length $|s|$

$$\phi(s) = \left(U(\mu k[T(e, n, \langle s \upharpoonright k \rangle)])\right)_{0 \leq n < |s|}$$

- issue is: $\phi(s)$ may not be well-defined: s may not be long enough to compute $\phi(s)(n)$ for all n up to $|s|$

- so we bound: define
  $b(s) = \max m < |s|$
  s.t. $\forall n \leq m \; \exists k < |s|$ s.t. $T(e, n, \langle s \upharpoonright k \rangle)$
  ($= 0$ if no such m)

- "Clearly" b is p.r. (all searches bounded in $|s|$)
- Now define
  $$\phi(s) = U(\mu k [T(e, n, \langle s \upharpoonright k \rangle)])_{0 \leq n < b(s)}$$
  output seq of length $b(s)$
  could be empty sequence!

- Observe: ϕ is
  ① monotone: if $s \subseteq t$, still have $\mu k\, T(e, n, \langle s \upharpoonright k \rangle) = \mu k\, T(e, n, \langle t \upharpoonright k \rangle)$ (searches for least witness)
  ② recursive (why?) ("clear" but convince yourself)

- also: $\alpha = \phi^*(\beta)$
  ↳ why: if k large enough s.t. $\forall i \leq n$ $\exists k_i < k$ s.t. $T(e, i, \bar{\beta}(k_i))$ ← i.e. $b(\beta \upharpoonright k) \geq n$
  then $\alpha \upharpoonright n \subseteq \phi(\beta \upharpoonright k)$
  i.e. can compute init seg of α of length at least n by plugging in $\beta \upharpoonright k$ ✓

such a k exists by Kleene, since α total

Corollary There is a sequence φ_e, $e \in \mathbb{N}$ of monotone recursive maps on $\mathbb{N}^{<\mathbb{N}}$ s.t. $\forall \alpha, \beta \in \mathbb{N}^{\mathbb{N}}$

$$\alpha \leq_T \beta \text{ iff } \exists e \text{ s.t. } \varphi_e^*(\beta) = \alpha$$

Pf: consider the maps constructed in above pf ✓

↳ can use the corollary to prove a basic fact about $(\mathcal{D}, \leq_T)$:

Theorem $\leq_T$ is not a linear order, i.e. $\exists \alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ s.t. $\alpha \not\leq_T \beta$ and $\beta \not\leq_T \alpha$

Pf: "diagonalize against the φ_e"

- we build α, β inductively as limits of $s^n, t^n \in \mathbb{N}^{<\mathbb{N}}$
- $\forall n$ we will have $|s^n| = |t^n| = l_n$ where $l_0 < l_1 < \ldots$ strictly incr. in $\mathbb{N}$
- and $\forall n$ $s^n \subseteq s^{n+1}$, $t^n \subseteq t^{n+1}$ (so $\alpha := \lim s^n$, $\beta := \lim t^n$ makes sense)

idea: @ odd stages $n = 2e+1$ pick s^n, t^n s.t. $\varphi_e(t^n)$ differs from s^n in someplace
↳ ensures $\varphi_e^*(\beta) \neq \alpha$
@ even stages $n = 2e$ pick so $\varphi_e(s^n)$ differs from t^n in some place
↳ $\varphi_e^*(\alpha) \neq \beta$