<u>Fact</u>: the jump is <u>not</u> a successor operation; in fact there are <u>many</u> degrees between $\underset{\sim}{a}$ and $\underset{\sim}{a}'$.

The arithmetic hierarchy

defined by $\exists k P(\vec{x}, k)$, P recursive

"$\exists$" Let Σ^0_1 := class of r.e. sets

"$\forall$" Π^0_1 := complements of r.e. sets ("co-r.e.")

Δ^0_1 : $\Sigma^0_1 \cap \Pi^0_1$ = recursive sets

117a: $A \in R$ iff $A, A^c \in RE$

<u>Note</u>: Σ^0_1 closed under $\exists$-quantification
hence Π^0_1 closed under $\forall$-quantification
Δ^0_1 closed under complements
(but neither $\forall$ nor $\exists$ quant'n in general)

i.e. family of relations defined by such fmlas

"$\exists\forall$" Can continue: $\Sigma^0_2 := \{\exists k P(\vec{x}, k) \mid P \in \Pi^0_1\}$

"$\forall\exists$" $\Pi^0_2 := \{P^c \mid P \in \Sigma^0_2\}$ i.e. $\neg P$

$\Delta^0_2 := \Sigma^0_2 \cap \Pi^0_2$

In general: $\Sigma^0_{n+1} := \{\exists k P(\vec{x}, k) \mid P \in \Pi^0_n\}$

$\Pi^0_{n+1} := \{\neg P \mid P \in \Sigma^0_{n+1}\}$

$\Delta^0_{n+1} := \Sigma^0_{n+1} \cap \Pi^0_{n+1}$

<u>Claim</u> $\Sigma^0_n \cup \Pi^0_n \subseteq \Delta^0_{n+1}$ $(= \Sigma^0_{n+1} \cap \Pi^0_{n+1})$

Pf: induction (you try)

↳ the collection of sets appearing in this hierarchy are called <u>arithmetical sets</u> (or <u>arithmetical relations</u>)

↳ so (by the claim) we have

class of arithmetical sets $= \bigcup_n \Sigma^0_n = \bigcup_n \Pi^0_n$
$= \bigcup_n \Delta^0_n$

<u>Prop'n</u> ① Σ^0_n is closed under $\wedge, \vee$, bdd quantification, substitution by total recursive functions, and $\exists$-quantification

② Π^0_n is closed under " "
" " "
" ", and $\forall$-quantification

③ Δ^0_n is closed under " "
" " "
" ", and $\neg$

<u>Pf:</u> Induction on n
- for $n=1$: did in 117a
- So sps for n.
- We prove ①. ②, ③ are similar.

<u>For $\wedge$:</u> Sps $R_1(\vec{x}) \Leftrightarrow \exists k P_1(k,\vec{x})$
$R_2(\vec{x}) \Leftrightarrow \exists l P_2(l,\vec{x})$

where P_1, P_2 are Π^0_n

- Define $P(k,l,\vec{x}) :\Leftrightarrow P_1(k,\vec{x}) \wedge P_2(l,\vec{x})$

- By induction $P \in \Pi^0_n$
- Hence $R_1 \wedge R_2 \in \Sigma^0_{n+1}$ since
  $(R_1 \wedge R_2)(\vec{x})$ iff $\exists w\, P((w)_0, (w)_1, \vec{x})$
  (i.e. the intersection of the rel'ns is definable by a Σ^0_{n+1} fmla)

- others are similar ✓

<u>Prop'n</u> ($\forall n$) Σ^0_n and Π^0_n have the enumeration property.

<u>Pf</u>. Induction on n.
- for $n=1 \Rightarrow 117a$.
- Sps true for n, and consider Σ^0_{n+1}
- want, for every $k \in \mathbb{N}$, a universal k-ary Σ^0_{n+1} relation $W^{k+1}(e, x_1, .., x_k)$

- by induction can find a universal $(k+1)$-ary Π^0_n rel'n $U^{k+2}(e, y, x_1, .., x_k)$

- define $W^{k+1}(e, x_1, .., x_k)$ by
  $$\exists y\, U^{k+2}(e, y, x_1, \ldots x_k)$$
- then $W^{k+1} \in \Sigma^0_{n+1}$ (by its def'n)
- moreover if $R \in \Sigma^0_{n+1}$ is k-ary then:
  $R(\vec{x})$ iff $\exists y\, Q(y, \vec{x})$
  for some $Q \in \Pi^0_n$
  iff $\exists y\, U^{k+2}(e, y, \vec{x})$
  for some $e \in \mathbb{N}$, by universality of U^{k+2}

iff $W^{k+1}(e, \vec{x})$

- hence W^{k+1} is universal for k-ary Σ^0_{n+1} rel'ns ✓
- similar for Π^0_{n+1} ✓✓

<u>Prop'n</u> Δ^0_n does <u>not</u> have the enumeration property

doesn't contradict Kleene enum'n thm? why?

Pf: you try (diagonal arg + fact that Δ^0_n closed under complements)

<u>Corollary</u>:

- Σ^0_n, Π^0_n not closed under complements
- Σ^0_n not closed under $\forall$
- Π^0_n not closed under $\exists$
- $\Sigma^0_n \cup \Pi^0_n \subsetneq \Delta^0_{n+1} \subsetneq \Sigma^0_{n+1}$, $\Delta^0_{n+1} \subsetneq \Pi^0_{n+1}$

<u>Pf</u>: exercise

<u>Example</u>: let $\{\varphi_e\}$ be an effective, acceptable enumeration of the unary recursive functions

have s-m-n functions

let $C = \{e \mid \varphi_e \text{ is total}\}$

what is complexity of C?