lang. represent multivariable polynomials w/ integer coefficients.)

will write $a<b$ instead of $<(a,b)$

- $0 = x_3$; $0 < S(0)$; $x^3 + \underline{2}z = y^2$ are all atomic fmlas
- $\forall x(0<x \Rightarrow \exists y(S(y)=x))$ is a non-atomic fmla.

↳ next want to define when a given fmla φ is T/F in a given structure

- not all fmlas intuitively have a truth value
- e.g. truth of "$0<x$" depends on value of x; but "$\exists x(0<x)$" is either T or F
- leads to concept of free and bound variables

__Def'n__ if $\exists x \varphi$ is a fmla, the __scope__ of the quantifier $\exists x$ is φ

__Def'n__ an occurrence of a variable x in a fmla φ is __bound__ if it occurs in a quantification $\exists x$ or in the scope of a quantifier in some subformula $\exists x \varphi$

otherwise the occurrence is __free__

- e.g. in $\forall y \exists z\, (y < x \wedge x < z)$ (all occurrences of) y, z are bound, x is free

- intuition: free variables are those we need to specify before a fmla becomes T or F.

Def'n a fmla φ is a sentence if it has no free variables

intuitively a sentence is T/F outright e.g. $\forall x \exists y\, (x<y)$

- we write $\varphi(x_1, .., x_n)$ to indicate that the free variables of φ are among $x_1, .., x_n$
  (some of the x_i may be bound or not occur in φ).

Semantics: first-order structures

Def'n for a language $L = \{R_i\} \cup \{f_j\} \cup \{c_k\}$, an L-structure is an object of the form

$$\mathcal{A} = \langle A, \{R_i^{\mathcal{A}}\}, \{f_j^{\mathcal{A}}\}, \{c_k^{\mathcal{A}}\} \rangle$$

where:
- A is a nonempty set
- each $R_i^{\mathcal{A}}$ is an n_i-ary relation on A (i.e. $R_i^{\mathcal{A}} \subseteq A^{n_i}$)
- each $f_j^{\mathcal{A}} : A^{m_j} \to A$ is an m_j-ary function on A

- each $c_k^{\mathcal{A}} \in A$ is an element of A

↳ we call the $R_i^{\mathcal{A}}, f_j^{\mathcal{A}}, c_k^{\mathcal{A}}$ the <u>interpretations</u> of the symbols R_i, f_j, c_k

↳ we often drop superscripts and confuse e.g. the actual relation $R^{\mathcal{A}}$ w/ the rel'n symbol R (notation abuse!)

<u>Evaluation of terms</u> given a term $t(x_1, .., x_n)$ we inductively define a function $t^{\mathcal{A}} : A^n \to A$:

vars appearing in t among $x_1, .., x_n$

- if $t(x_1, .., x_n)$ is x_i then $t^{\mathcal{A}}(a_1, .., a_n) = a_i$
- if $t(x_1, .., x_n)$ is c then $t^{\mathcal{A}}(a_1, .., a_n) = c^{\mathcal{A}}$
- if $t(x_1, .., x_n)$ is $f(t_1(\vec{x}), .., t_m(\vec{x}))$ then $t^{\mathcal{A}}(\vec{a}) = f^{\mathcal{A}}(t_1^{\mathcal{A}}(\vec{a}), .., t_m^{\mathcal{A}}(\vec{a}))$

e.g. if $L = \{<, s, +, \cdot, 0\}$ and $\mathcal{N} = \langle \mathbb{N}, <^{\mathcal{N}}, s^{\mathcal{N}}, +^{\mathcal{N}}, \cdot^{\mathcal{N}}, 0^{\mathcal{N}} \rangle$ where

$<^{\mathcal{N}}$ is usual $<$ rel'n on $\mathbb{N}$

$s^{\mathcal{N}}$ is usual successor function on $\mathbb{N}$

etc.

then if $t(x_1, x_2)$ is the term $\underline{3}\, x_1^2 + \underline{2}\, x_2 + \underline{5}$ then

$t^{\mathcal{N}}(10, 7) = 3 \cdot 10^2 + 2 \cdot 7 + 5 = 319$

- now let $\mathcal{M}$ be the structure in the same language w/ all the same interpretations except $+^{\mathcal{M}}$ is interpreted as the exp function:

$$n +^{\mathcal{M}} m = n^m$$

- then $t^{\mathcal{M}}(10,7) = ((3 \cdot 10^2)^{2 \cdot 7})^5$

assuming parentheses are:

$$(3x_1^2 + 2x_2) + 5$$

Truth valuations of formulas:
- Fix L and an L-structure $\mathcal{A}$
- want to iteratively define when a fmla is true in $\mathcal{A}$ (w/ free vars specified)

- Sps $\varphi(x_1, .., x_k) = \varphi(\vec{x})$ is a fmla
- Given $\vec{a} = (a_1, .., a_k)$, inductively define when φ holds in $\mathcal{A}$ w/ variable assignment $x_i \to a_i$

write: $\mathcal{A}, \vec{a} \models \varphi$, or $\mathcal{A} \models \varphi(\vec{a})$

(i) φ is $R(t_1, .., t_k)$

$\mathcal{A}, \vec{a} \models \varphi$ iff $R^{\mathcal{A}}(t_1^{\mathcal{A}}(\vec{a}), .., t_k^{\mathcal{A}}(\vec{a}))$

(ii) φ is $t_1 = t_2$

$\mathcal{A}, \vec{a} \models \varphi$ iff $t_1^{\mathcal{A}}(\vec{a}) = t_2^{\mathcal{A}}(\vec{a})$

(iii) φ is $\neg\psi$

$\mathcal{A}, \vec{a} \models \varphi$ iff $\mathcal{A}, \vec{a} \not\models \psi$

over →

(iv) φ is $\psi \wedge \tau$

$\mathcal{A}, \vec{a} \models \varphi$ iff $\mathcal{A}, \vec{a} \models \psi$ and $\mathcal{A}, \vec{a} \models \tau$

(v) φ is $\exists x_i\, \psi(x_1, \ldots, x_i, \ldots, x_k)$

$\mathcal{A}, \vec{a} \models \varphi$ iff there is $b \in A$ s.t.

$\mathcal{A}, (a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_k) \models \psi$

<u>Ex</u> Consider $L = \{<, S, +, \cdot, 0\}$
and $\mathcal{N} = \langle \mathbb{N}, <, S, \cdot, 0 \rangle$ (all symbols interp'd in usual way)

- then $\mathcal{N} \models$ "there are infinitely many primes"

i.e.

$$\mathcal{N} \models \forall x \exists y \, [\underline{1} < y \wedge x < y \wedge \underbrace{\forall z\, (\exists w (w \cdot z = y) \Rightarrow (z = \underline{1} \vee z = y))}_{\text{"y is prime"}}]$$

but $\mathcal{N} \not\models$ "4 is prime" i.e. $\mathcal{N}, 4 \not\models$ "y is prime"

- Now let $\mathcal{M} = \langle \mathbb{N}, <, 10^{(\cdot)}, +, \cdot, 0 \rangle$
i.e. same as $\mathcal{N}$ except we interpret $S(\cdot)$ as $10^{(\cdot)}$ (i.e. $S^{\mathcal{M}}(n) = 10^n$).
- Let τ denote the sentence
$\forall x \forall y\, (S(x) + y) = S(x+y))$
- then $\mathcal{N} \models \tau$ since $x + 1 + y = x + y + 1$
- but $\mathcal{M} \not\models \tau$ since $10^x + y \neq 10^{x+y}$ in general.

<u>Def'n</u> - if φ is a sentence in lang. of $\mathcal{A}$, read $\mathcal{A} \models \varphi$ as "$\mathcal{A}$ models φ"
- a set of sentences Σ is called a theory
- write $\mathcal{A} \models \Sigma$ if $\mathcal{A} \models \varphi$ for all $\varphi \in \Sigma$.

<u>Ex</u>: let $L = \{\cdot, e\}$ (binary function symbol; constant)
Consider Σ consisting of following sentences

$$\forall x \forall y \forall z \; ((x \cdot y) \cdot z = x \cdot (y \cdot z))$$
$$\forall x \; (x \cdot e = x \;\wedge\; e \cdot x = x)$$
$$\forall x \exists y \; (x \cdot y = e \;\wedge\; y \cdot x = e)$$

Then: given an L-structure $\mathcal{G} = \langle G, \cdot^{\mathcal{G}}, e^{\mathcal{G}} \rangle$ have $\mathcal{G} \models \Sigma$ iff $\mathcal{G}$ is a group.

<u>Def'n</u> For a theory Σ and a sentence φ, we write $\Sigma \models \varphi$ if whenever $\mathcal{A} \models \Sigma$ then $\mathcal{A} \models \varphi$.
Say Σ <u>logically</u> <u>implies</u> φ

"semantically implies" might be better

e.g. if φ is $e \cdot e = e$ and Σ is as above then $\Sigma \models \varphi$ (since $e \cdot e = e$ is true in every group)

<u>Def'n</u> – a <u>tautology</u> is a sentence φ s.t. $\Sigma = \emptyset$ <u>logically</u> implies φ

↳ i.e. φ is true in every structure in the language

↳ write: $\models \varphi$ ↳ say: φ is <u>logically valid</u>

– Two sentences are <u>logically equivalent</u> if $\varphi \Leftrightarrow \psi$ is a tautology

– e.g. $c = c$ and $\forall x \forall y\, R(x,y) \Leftrightarrow \forall y \forall x\, R(x,y)$ are tautologies

(hence $\forall x \forall y\, R(x,y)$ logically equiv to $\forall y \forall x\, R(x,y)$)

– we say that a formula $\varphi(x_1, \ldots, x_k)$ (possibly w/ free variables) is a tautology if $\forall x_1 \ldots \forall x_k\, \varphi(\vec{x})$ is a tautology

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<u>Quantifier depth</u>

– a useful way to measure complexity of a fmla: count # of $\forall\exists$ alternations at beginning (think: arithmetic hierarchy)

– need a way of pulling out quantifiers to front for this to make sense in general

"Q.F."

<u>Def'n</u> – a fmla $\tau(\vec{y})$ is <u>quantifier free</u> if it contains no $\forall$'s or $\exists$'s.

"p.n.f"

- a fmla is in **prenex normal form** if it looks like

$Q_1 y_1 \ldots Q_n y_n \, \tau(\vec{y})$

where each Q_i is $\forall$ or $\exists$ and τ is quantifier free

e.g. $\forall x \exists y (x < y)$ is in p.n.f. but $\forall x (0 < x \Rightarrow \exists y (s(y) = x))$ is not.

Prop'n every fmla (in any lang L) is logically equiv to a fmla in p.n.f.

Pf: idea: "keep pulling out quantifiers"
only issue: sometimes need to replace variables.

e.g. $\exists x P(x) \wedge \exists x Q(x)$
is equiv to $\exists x \exists y (P(x) \wedge Q(y))$
but not to $\exists x (P(x) \wedge Q(x))$ in general.

- Following logical equivalences all we need:

$\neg \exists x \varphi$	equiv to	$\forall x \neg \varphi$
$\neg \forall x \varphi$	"	$\exists x \neg \varphi$
$\varphi \wedge \exists x \chi$	"	$\exists x (\varphi \wedge \chi)$ if no x in φ
$\exists x \varphi \wedge \chi$	"	$\exists x (\varphi \wedge \chi)$ if no x in χ

... etc. for $\forall$ and $\vee$