Powersets

- Consider the set $A = \{1,2,3\}$
- Can we list all the subsets of A?

Sure:

$$\begin{array}{cccc} & \{1\} & \{1,2\} & \\ \emptyset & \{2\} & \{1,3\} & \{1,2,3\} \\ & \{3\} & \{2,3\} & \end{array}$$

- the set of these subsets is called the <u>powerset</u> of A:

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

<u>Def'n</u> Given a set X, the <u>powerset of X</u>, denoted $P(X)$, is the set of all subsets of X,

i.e. $Y \in P(X)$ iff $Y \subseteq X$.

- powersets are typically big!
Hard to write down (like we did w/ A) in general

Ex: ① - $\{1, 91, 10^{100}\} \subseteq \mathbb{N}$

hence $\{1, 91, 10^{100}\} \in P(\mathbb{N})$

- also, $E \in P(\mathbb{N})$, $O \in P(\mathbb{N})$, $\mathbb{N} \in P(\mathbb{N})$

Since these are all subsets of $\mathbb{N}$

- but $\{-1, 0, 1\} \notin P(\mathbb{N})$ since $\{-1, 0, 1\} \not\subseteq \mathbb{N}$

② Prop'n: For any set X, we have:

ⓘ $\emptyset \in P(X)$ ⓘⓘ $X \in P(X)$

Pf: we know $\emptyset \subseteq X$ and $X \subseteq X$. ✓

③ Prop'n: For any sets A, B, if $A \subseteq B$ then $P(A) \subseteq P(B)$.

Pf: - Assume $A \subseteq B$ and fix $X \in P(A)$

- then $X \subseteq A$, by def'n of powerset.
- Since $A \subseteq B$ we have $X \subseteq B$ by transitivity of $\subseteq$ (proved before)
- Hence $X \in P(B)$.
- Since X was arbitrary, we have $P(A) \subseteq P(B)$ ✓

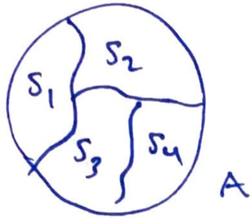
④ What is $P(\emptyset)$?

- only subset of $\emptyset$ is $\emptyset$
- hence $P(\emptyset) = \{\emptyset\}$

Partitions

A <u>partition</u> of a set A is a collection of subsets of A that split up A into <u>disjoint</u> pieces.

<u>Picture</u>:



$\{S_1, S_2, S_3, S_4\}$ is a partition of A into 4 pieces.

<u>Formal def'n</u>: Let A be a set. A <u>partition of A</u> is a set P of sets s.t.

① for every $X \in P$ we have $X \subseteq A$ and $X \neq \emptyset$

② for every $X, Y \in P$, if $X \neq Y$ then $X \cap Y = \emptyset$

③ $\bigcup_{X \in P} X = A$

<u>vocab</u>: ② says that the sets in the partition P are "pairwise disjoint"

Ex's ① Let $A = \{1,2,3,4,5,6\} = [6]$

Let $S_1 = \{1\}$

$S_2 = \{2,3,6\}$

$S_3 = \{4,5\}$

Then $P = \{S_1, S_2, S_3\} = \{\{1\}, \{2,3,6\}, \{4,5\}\}$ is a partition of A.

Why: ① S_1, S_2, S_3 all nonempty subsets of A. ✓

② $S_1 \cap S_2 = S_1 \cap S_3 = S_2 \cap S_3 = \emptyset$

③ $S_1 \cup S_2 \cup S_3 = A$.

However: - $\{S_1, S_2\}$ isn't a partition of A since ③ fails.

- Let $S_4 = \{1, 6\}$. Then $\{S_1, S_2, S_3, S_4\}$ isn't a partition either since e.g. $S_1 \cap S_4 = \{1\} \neq \emptyset$ so ② fails.

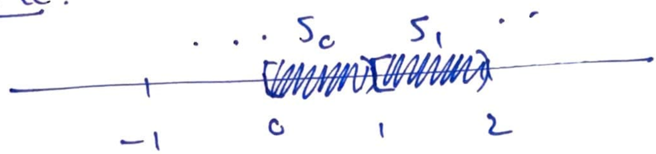
② $\{E, O\}$ is a partition of $\mathbb{N}$.

3 - Consider $\mathbb{R}$

- for every $z \in \mathbb{Z}$, define

$$S_z = \{x \in \mathbb{R} \mid z \le x < z+1\}$$
$$= [z, z+1)$$

Pic:



Then $\{S_z : z \in \mathbb{Z}\}$ is a partition of $\mathbb{Z}$

$$= \{\dots S_{-1}, S_0, S_1, S_2, \dots\} \quad \text{(why?)}$$

"on the other hand"

4 OTOH if we define:

$$T_z = \{x \in \mathbb{R} \mid z \le x \le z+1\}$$
$$= [z, z+1]$$

then $\{T_z : z \in \mathbb{Z}\}$ is <u>not</u> a partition of $\mathbb{R}$ since e.g.

$$T_1 \cap T_2 = [1,2] \cap [2,3] = \{2\} \neq \emptyset$$

Cartesian Products

Def'n: Suppose A and B are sets. The Cartesian Product of A, B, denoted $A \times B$, is the set of all ordered pairs (a,b) with $a \in A$ and $b \in B$.

i.e. $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$

illegal use of set-builder notation, but can be formalized.

Note: - for ordered pairs, order is important! First coord. from A, second coord. from B, and often $A \neq B$.

- we sometimes write A^2 for $A \times A$.

Ex's ① if $A = \{1,2,3\}$ $B = \{*, \heartsuit\}$

then $A \times B = \{(1,*), (2,*), (3,*), (1,\heartsuit), (2,\heartsuit), (3,\heartsuit)\}$

② $(1,2) \in \mathbb{N} \times \mathbb{N}$ and $(1,2) \in \mathbb{N} \times \mathbb{R}$, also $(1,\sqrt{2}) \in \mathbb{N} \times \mathbb{R}$ but $(1,\sqrt{2}) \notin \mathbb{N} \times \mathbb{N}$ since $\sqrt{2} \notin \mathbb{N}$.

(3) <u>Triples</u>: Given sets A, B, C can define $A \times B \times C = \{(a,b,c) \mid a \in A, b \in B, c \in C\}$

e.g. $(1, \pi, 2+i) \in \mathbb{N} \times \mathbb{R} \times \mathbb{C}$.

↳ $A \times B \times C$ not strictly the same as $(A \times B) \times C$: el'ts of first set look like (a,b,c), el'ts of second $((a,b),c)$

↳ but, these sets are "essentially" the same.

(4) Can use pairs, triples, etc. in set-builder notation, e.g. let

$$X = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m \in E \text{ and } n \in O\}$$

then $(2,5) \in X$ but $(1,6) \notin X$.

Proofs with sets

<u>Proving $A \subseteq B$</u>:
1. Fix an arbitrary $a \in A$
2. Prove $a \in B$
3. Conclude, since $a \in A$ was arbitrary, that for <u>every</u> $x \in A$ we have $x \in B$, i.e. $A \subseteq B$.

Ex ① Prop'n: Suppose A, B, X are sets. If $X \subseteq A$ and $X \subseteq B$ then $X \subseteq A \cap B$.

Pf: - fix $x \in X$
- since $X \subseteq A$ we have $x \in A$
- since $X \subseteq B$ we have $x \in B$
- hence $x \in A \cap B$
- since $x \in X$ was arbitrary, we have $X \subseteq A \cap B$.

② Prop'n Sps A, B are sets.
Then: $P(A) \cap P(B) \subseteq P(A \cap B)$

Pf: - fix $X \in P(A) \cap P(B)$
- then $X \in P(A)$ and $X \in P(B)$, by def'n of $\cap$
- hence $X \subseteq A$ and $X \subseteq B$, by def'n of powerset
- hence by our previous prop'n we have $X \subseteq A \cap B$, i.e. $X \in P(A \cap B)$
- since $X \in P(A) \cap P(B)$ was arbitrary we have $P(A) \cap P(B) \subseteq P(A \cap B)$