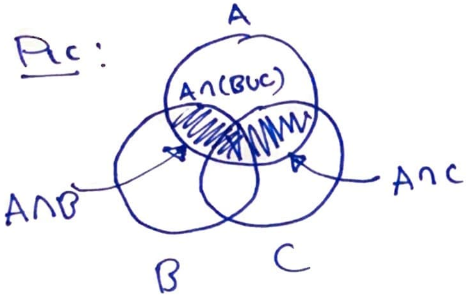
Proving $A = B$

1. Prove $A \subseteq B$
2. Prove $B \subseteq A$
3. Conclude $A = B$.

Ex: 1 Prop'n For any sets A, B, C we have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Pic:



Pf: ($\subseteq$) - fix $x \in A \cap (B \cup C)$

- then: (i) $x \in A$

and (ii) $x \in B \cup C$

case ii.1: $x \in B$. Then since $x \in A$ as well we have $x \in A \cap B$ in this case.

case ii.2: $x \in C$. Then since $x \in A$ we have $x \in A \cap C$ in this case.

Hence either $x \in A \cap B$ or $x \in A \cap C$, i.e. $x \in (A \cap B) \cup (A \cap C)$.

since x was arbitrary we have
$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

($\supseteq$) - fix $x \in (A \cap B) \cup (A \cap C)$

Case 1: $x \in A \cap B$.
- then $x \in A$ and $x \in B$
- hence $x \in A$ and ($x \in B$ or $x \in C$)
- i.e. $x \in A \cap (B \cup C)$

Case 2: $x \in A \cap C$
- then $x \in A$ and $x \in C$
- hence $x \in A$ and ($x \in B$ or $x \in C$)
- i.e. $x \in A \cap (B \cup C)$

- hence in all cases we have $x \in A \cap (B \cup C)$
- since x was arbitrary we have
$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ ✓

$\boxed{\text{Counterexamples}}$ To show that a (universal) statement is false, sufficient to provide a single counterexample.

Ex: Disprove the following:

Claim: For any sets A, B, C
if $A \subseteq B \cup C$ then either $A \subseteq B$ or $A \subseteq C$.

Sol'n: Consider the sets

$A = \{2,3\}$
$B = \{1,2\}$ then: $A = \{2,3\} \subseteq \{1,2,3,4\}$
$= B \cup C$
$C = \{3,4\}$

but $A \not\subseteq B$ $A \not\subseteq C$
Hence the claim is false.

Using Contradiction

to prove that a statement S is true, you can:

① Assume S is false

② Show that this assumption contradicts your hypothesis or a previously proved statement.

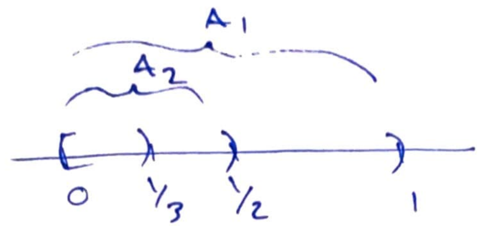
③ Conclude S is true.

Ex: Prop'n. For every $n \in \mathbb{N}$ define the set

$$A_n = \{x \in \mathbb{R} \mid 0 \leq x < \tfrac{1}{n}\} = [0, \tfrac{1}{n})$$

Then we have: $\bigcap_{n \in \mathbb{N}} A_n = \{0\}$

Picture:



Pf: ($\supseteq$) Proving $\{0\} \subseteq \bigcap_{n \in \mathbb{N}} A_n$ is equivalent to proving $0 \in \bigcap_{n \in \mathbb{N}} A_n$, i.e. that for every $n \in \mathbb{N}$, we have $0 \in A_n$.

- So fix $n \in \mathbb{N}$.
- Since $A_n = \{x \in \mathbb{R} \mid 0 \leq x < \frac{1}{n}\}$ we clearly have $0 \in A_n$, since indeed $0 \leq 0 < \frac{1}{n}$.
- Since $n \in \mathbb{N}$ was arbitrary, we have $0 \in A_n$ for every n, i.e. $0 \in \bigcap_{n \in \mathbb{N}} A_n$

hence $\{0\} \subseteq \bigcap_{n \in \mathbb{N}} A_n$ ✓

($\subseteq$) To prove $\bigcap_{n\in\mathbb{N}} A_n \subseteq \{0\}$ we argue by contradiction.

- Sps (toward a contradiction) that $\bigcap_{n\in\mathbb{N}} A_n \not\subseteq \{0\}$
- then there is some $x \in \bigcap_{n\in\mathbb{N}} A_n$ s.t. $x \notin \{0\}$, i.e. $x \neq 0$.
- since in particular $x \in \underbrace{A_1}_{=[0,1)}$, but $x \neq 0$, we must have $0 < x < 1$.

- Pick an $N \in \mathbb{N}$ large enough so that $\frac{1}{N} < x$ (always possible - why?)

- but then $x \notin \underbrace{A_N}_{=[0,\frac{1}{N})}$
- hence $x \notin \bigcap_{n\in\mathbb{N}} A_n$
- a contradiction, as we had $x \in \bigcap_{n\in\mathbb{N}} A_n$

- Hence our assumption $\bigcap_{n\in\mathbb{N}} A_n \not\subseteq \{0\}$ is <u>false</u>. (35)

- Hence $\bigcap_{n\in\mathbb{N}} A_n \subseteq \{0\}$.

- Combining w/ above yields $\bigcap_{n\in\mathbb{N}} A_n = \{0\}$.