Ch. 4: Intro to Mathematical Logic (36)

<u>Goals</u>: - Learn how to write statements more formally (more symbols, fewer words)
- see how the <u>form</u> of a statement can suggest the form of its proof.

Recall: <u>Def'n</u> (informal) A <u>mathematical statement</u> (or <u>prop'n</u>) is a grammatically correct declarative sentence that is either <u>true</u> or <u>false</u>.

↳ can consist of words and/or symbols
↳ "statement" can be rigorously defined, but need formal logic
↳ "grammatically correct" also has a precise meaning in that context

<u>Ex's</u> ① Every integer is a real number (T)
② Every real number is an integer (F)
③ There exists $x \in \mathbb{R}$ s.t. $x \notin \mathbb{Z}$ (T)

(4) $1+1=2$ (T)

(5) There are infinitely many twin primes (unknown ... but either T or F)

<u>Nonex's</u>: (1) $\{ ! \pi *$ (grammatically incorrect/meaningless)

(2) Shakespeare (not declarative/ no truth value)

(3) $x^2+1=2$

↳ a meaningful sequence of symbols asserting an equality, but no truth value ... unless x is specified!

- called a <u>variable proposition</u>: a sentence that becomes a statement once its variables are specified (or quantified over ... more later)

- will use $P, Q, R, \ldots$ to denote statements and $P(x), Q(x,y), \ldots$ for var. prop'n

e.g. might say: - Let P denote "$5^2+1=2$" (F)

- Let $Q(x)$ denote "$x^2+1=2$"

Then $Q(5)$ is the statement
"$5^2+1=2$" (F)

$Q(1)$ is:
"$1^2+1=2$" (T)

<u>More var prop'ns</u>: ① $x^2+1 \le 0$
② $x \in \mathbb{Z}$ and $x^2 < 39$
③ $z = x+y$

note: you should indicate when abbreviating a var. prop'n w/ multiple variables, e.g. could use $Q(x,y,z)$ for ③.

Then: $Q(1,2,3)$ is "$3=1+2$" (T)
$Q(3,2,1)$ is "$1=3+2$" (F)

$\boxed{\text{Quantifiers}}$ the other way to turn a var. prop'n into a statement is to <u>quantify</u> over its variables.

<u>Ex</u>: "$x^2+1=2$" is a var. prop'n, but "there exists $x \in \mathbb{R}$ s.t. $x^2+1=2$" (T) is a statement, as is

"for every $x \in \mathbb{R}$ we have $x^2+1=2$" (F)

The clauses "There exists $x \in S, \ldots$"
"For every $x \in S, \ldots$"

are two types of <u>quantification</u> of the variable x.

- we'll use the symbols

$\forall$ (read: "for all")

$\exists$ (read: "there exists")

the "universal quantifier" the "existential quantifier"

- Given a var. prop'n $P(x)$ and a set S, "For all $x \in S$ we have $P(x)$"

and "There exists $x \in S$ s.t. $P(x)$"

are <u>statements</u>.

- we'll denote these by

$(\forall x \in S)\ P(x)$ (book uses:

$(\exists x \in S)\ P(x)$ $\forall x \in S.\ P(x)$

$\exists x \in S.\ P(x)$)

respectively

Ex's: ① $(\exists x \in \mathbb{N})(x<5)$

"There exists $x \in \mathbb{N}$ s.t. $x<5$" (T)

② $(\forall x \in \mathbb{N})(x<5)$

"For every $x \in \mathbb{N}$ we have $x<5$" (F)

③ $(\forall x \in \mathbb{N})(0<x)$ (T)

④ $(\forall x \in \mathbb{R})(0<x)$ (F)

Multiple Quantifiers

⑤ Consider the var. prop'n "$x+y \geq 2$"

Then: $(\forall y \in \mathbb{N})(x+y \geq 2)$ is still a var. prop'n, but now with only one "free variable" x. But

$$(\forall x \in \mathbb{N})(\forall y \in \mathbb{N})(x+y \geq 2)$$ is a

statement (T).

↳ can also write $(\forall x, y \in \mathbb{N})(x+y \geq 2)$

"For all $x, y \in \mathbb{N}$ we have $x+y \geq 2$"

⑥ Can also mix $\forall$s and $\exists$s, but then beware: order of quantifiers becomes important!

e.g. $(\forall x \in \mathbb{N})(\exists y \in \mathbb{R})(y^2 = x)$

"For every $x \in \mathbb{N}$, there exists $y \in \mathbb{R}$ s.t. $y^2 = x$" asserts every natural number has a real square root (true.)

whereas if we reverse the quantifiers:

(7) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{N})(y^2 = x)$

asserts there exists a real number y s.t. every natural number x is equal to y^2.

- a perfectly well-written statement, but definitely a false one.

- moral: order of quantifiers makes a big diff!

We can also have "inside quantifiers"

e.g. (8) $(\forall x \in \mathbb{N})(x > 0 \text{ and } (\exists y \in \mathbb{N})(y > x))$

(9) $(\forall x \in \mathbb{R})(\text{if } x \geq 0 \text{ then } (\exists y \in \mathbb{R})(y^2 = x))$

are both legit statements (and both (T))

Note: we insist all quantified variables range over a specified set.