$A \cap B = \{x \in U \mid (x \in A) \wedge (x \in B)\}$
$A \cup B = \{x \in U \mid (x \in A) \vee (x \in B)\}$
$\overline{A} = \{x \in U \mid \neg(x \in A)\}$

equiv to writing "$x \notin A$"

- we'll explore connections between connectives and set operations were later.

Implication Given statements P, Q the statement $P \Rightarrow Q$ is read "if P, then Q" or "P implies Q"

$P \Rightarrow Q$ is true iff

wherever P is true, Q is also true

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

notice: - $P \Rightarrow Q$ is always true when P is false (often a confusing point)
- $P \Rightarrow Q$ only false when P is true and Q is false

- statements of the form $P \Rightarrow Q$ are called "conditional statements"

<u>Ex's</u>

1. "$\underset{T}{(1+1=2)} \Rightarrow \underset{T}{(1+1+1=3)}$" is <u>true</u>

2. "$\underset{T}{(1+1=2)} \Rightarrow \underset{F}{(1+1+1=4)}$" is <u>false</u>

3. "$(1+1=2) \Rightarrow (\sqrt{2} \notin \mathbb{N})$" is <u>true</u>

even though P, Q in this example are apparently unrelated statements.

4. "My name is Sally $\Rightarrow$ My name begins with S" is <u>true</u>

↳ both premise P and conclusion Q are <u>false</u>, therefore (by def'n) $P \Rightarrow Q$ is <u>true</u> ("false $\Rightarrow$ false" is (T))

5. "Tomorrow is Sunday $\Rightarrow$ My name is Garrett" is also <u>true</u> ("false $\Rightarrow$ true" is (T))

6. $(\exists x \in \mathbb{R})(x^2 = -1) \Rightarrow (1+1=3)$

is true: automatically since premise is false, though unrelated to conclusion

7. can also use $\Rightarrow$ in var. prop'ns

e.g. $x \geq 2 \Rightarrow x^2 \geq 4$

is a var. prop'n and

$(\forall x \in \mathbb{R})(x \geq 2 \Rightarrow x^2 \geq 4)$

is a true statement, since

for every $x \in \mathbb{R}$

either $x \geq 2$ is true,
in which case $x^2 \geq 4$ is also true
and "true $\Rightarrow$ true" is true

or $x \geq 2$ is false,
in which case $x \geq 2 \Rightarrow x^2 \geq 4$
is true automatically

i.e. for every $x \in \mathbb{R}$, "$(x \geq 2) \Rightarrow (x^2 \geq 4)$"
is (T), i.e. $(\forall x \in \mathbb{R})(x \geq 2 \Rightarrow x^2 \geq 4)$
is (T)

⑧ OTOH: $(\forall x \in \mathbb{R})(x^2 \geq 4 \Rightarrow x \geq 2)$ 50

is false since there is a real number x (e.g. $x = -3$) s.t. $x^2 \geq 4$ is (T) but $x \geq 2$ is (F).

$\boxed{\text{Equivalence}}$ Given statements P, Q the statement $P \Leftrightarrow Q$

(read: "P if and only if Q")

is true iff P, Q have the same truth values

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Ex's ① $(1+1=2) \Leftrightarrow (1+1+1=3)$ is (T)

② $(1+1=3) \Leftrightarrow (1+1+1=4)$ is (T)

③ $(\forall x \in \mathbb{N})(n > 0) \Leftrightarrow (1+1=2)$ is (T)

P, Q need not be "related"

④ $(1+1=2) \Leftrightarrow (2+2=5)$ is (F)

Can also use $\Leftrightarrow$ in var prop'ns, e.g.

$$(x \geq 0) \Leftrightarrow (\exists y \in \mathbb{R})(y^2 = x)$$

is a legit var. prop'n

and $(\forall x \in \mathbb{R})\left[(x \geq 0) \Leftrightarrow (\exists y \in \mathbb{R})(y^2 = x)\right]$

is a true statement, since:

for every $x \in \mathbb{R}$ the ~~stmt~~ statements "$x \geq 0$" and "$(\exists y \in \mathbb{R})(y^2 = x)$" are either both true or both false.

Def'n statements P, Q are said to be logically equivalent iff they have the same truth value, i.e. iff $P \Leftrightarrow Q$ is true.

- e.g. $1+1=2$ and $1+1+1=3$ are logically equiv.
- will be most interested in logically equivalent forms for connected (esp. negated) and quantified statements.

Negating Quantified statements

- Sps $P(x)$ is a variable prop'n and S is a set.

- Consider the negated statements

① $\neg (\forall x \in S)\, P(x)$

② $\neg (\exists x \in S)\, P(x)$

Observe ① is true iff there is an $x \in S$ s.t. $P(x)$ is false, i.e. iff

$(\exists x \in S) \neg P(x)$ is true

② is true iff for all $x \in S$ we have $P(x)$ is false, i.e.

$(\forall x \in S) \neg P(x)$ is true.

This shows

$$\neg (\forall x \in S)\, P(x) \iff (\exists x \in S) \neg P(x)$$

is always true (regardless of $P(x)$)

i.e. that $\neg (\forall x \in S)\, P(x)$ and $(\exists x \in S) \neg P(x)$ are logically equivalent.

Likewise

$\neg (\exists x \in S)\, P(x)$ and $(\forall x \in S) \neg P(x)$

are logically equiv.

– these equivalences often useful when trying to prove quantified statements by contradiction.

Ex's ① $\neg(\forall x \in \mathbb{R})(x \in \mathbb{N})$ "not all reals are naturals"

is equiv. to

$(\exists x \in \mathbb{R}) \neg(x \in \mathbb{N})$ "there is a real which is not a natural"

both are (T)

(note: we'll often write $\neg(x \in \mathbb{N})$ as $x \notin \mathbb{N}$, $\neg(x = y)$ as $x \neq y$, etc.)

② $\neg(\exists x \in \mathbb{R})(x+1=0)$

is equiv to

$(\forall x \in \mathbb{R})(x+1 \neq 0)$

(both (F))

③ For multiple quantifiers, iterate the process:

$\neg(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy=1) \rightarrow$ "not every real has a mult. inverse"

equiv. to: $(\exists x \in \mathbb{R}) \neg(\exists y \in \mathbb{R})(xy=1)$

also: $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(xy \neq 1)$

(both are (T): 0 has no multiplicative inverse)

→ "there is a real w/o a multiplicative inverse"