② $(\forall x \in \mathbb{N})[(x \in O) \Rightarrow (x+1 \in E)]$

equiv to: $(\forall x \in \mathbb{N})[(x \notin O) \vee (x+1 \in E)]$

also equiv to: $(\forall x \in \mathbb{N})[(x+1 \notin E) \Rightarrow (x \notin O)]$

(all (T))

③ Let $P = \{2,3,5,7,\ldots\}$ denote the set of primes. Then:

$(\forall x \in \mathbb{N})[(x \in P) \Leftrightarrow (x \in O)]$

equiv to:

$(\forall x \in \mathbb{N})[((x \in P) \Rightarrow (x \in O)) \wedge ((x \in O) \Rightarrow (x \in P))]$

(both (F))

④ Consider the following (true) statement: $(\forall x \in \mathbb{R})[(x \geq 0) \Leftrightarrow (\exists y \in \mathbb{R})(y^2 = x)]$

Let's put its negation in positive form:

$\neg(\forall x \in \mathbb{R})[(x \geq 0) \Leftrightarrow (\exists y \in \mathbb{R})(y^2 = x)]$

$\Leftrightarrow (\exists x \in \mathbb{R})\neg[(x \geq 0) \Leftrightarrow (\exists y \in \mathbb{R})(y^2 = x)]$

$\Leftrightarrow (\exists x \in \mathbb{R})[((x \geq 0) \wedge \neg(\exists y \in \mathbb{R})(y^2 = x)) \vee$

$(\neg(x \geq 0) \wedge (\exists y \in \mathbb{R})(y^2 = x))]$

$\Leftrightarrow (\exists x \in \mathbb{R}) [((x \geq 0) \wedge (\forall y \in \mathbb{R})(y^2 \neq x)) \vee$
$((x<0) \wedge (\exists y \in \mathbb{R})(y^2 = x))]$

logically equiv. to original negated statement (and false)

More useful equivalences

Prop'n The following equivalences hold:

① $P \wedge (Q \wedge R)$	$\Leftrightarrow$	$(P \wedge Q) \wedge R$	Associative laws
② $P \vee (Q \vee R)$	$\Leftrightarrow$	$(P \vee Q) \vee R$	
③ $P \wedge (Q \vee R)$	$\Leftrightarrow$	$(P \wedge Q) \vee (P \wedge R)$	Distributive laws.
④ $P \vee (Q \wedge R)$	$\Leftrightarrow$	$(P \vee Q) \wedge (P \vee R)$	

Pf: try the truth ~~tables~~ tables!

Proving equality of sets using $\Leftrightarrow$

There is a strong analogy between our logical connectives and the set operations from Ch. 3:

Connective	operation
$P \wedge Q$	$A \cap B$
$P \vee Q$	$A \cup B$
$P \Rightarrow Q$	$A \subseteq B$
$P \Leftrightarrow Q$	$A = B$
$\neg P$	$\overline{A}$

more precisely: observe that the statements "$x \in A \cap B$"
"$x \in A \cup B$" etc.

are equiv. to "$x \in A \ \wedge \ x \in B$"
"$x \in A \ \vee \ x \in B$" etc.

Can use this observation to prove set identities in a new way, using $\Leftrightarrow$'s.

Theorem Sps A, B are sets and $\mathcal{U}$ is a universal set with $A, B \subseteq \mathcal{U}$.
Then we have:

① $\overline{\overline{A}} = A$

② $\overline{A \cap B} = \overline{A} \cup \overline{B}$

③ $\overline{A \cup B} = \overline{A} \cap \overline{B}$

(looks like:
$\neg\neg P \Leftrightarrow P$
$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

Pf: ① Fix $x \in U$ ⟵ not in A or $\bar{A}$!!!

then $x \in \bar{\bar{A}} \Leftrightarrow x \notin \bar{A}$ (def'n of complement

$\Leftrightarrow \neg(x \in \bar{A})$

$\Leftrightarrow \neg(\neg(x \in A))$ (def'n of comp)

$\Leftrightarrow x \in A$ $(\neg\neg P \Leftrightarrow P)$

This chain of equivalencies shows:

$(x \in \bar{\bar{A}}) \Leftrightarrow (x \in A)$

i.e. $(x \in \bar{\bar{A}}) \Rightarrow (x \in A)$ (shows: $\bar{\bar{A}} \subseteq A$)

and $(x \in A) \Rightarrow (x \in \bar{\bar{A}})$ (shows $A \subseteq \bar{\bar{A}}$)

hence we've proved $\bar{\bar{A}} = A$.

② Fix $x \in U$.

then: $x \in \overline{A \cap B} \Leftrightarrow \neg(x \in A \cap B)$ def'n of comp.

$\Leftrightarrow \neg(x \in A \wedge x \in B)$ def'n of $\cap$

$\Leftrightarrow \neg(x \in A) \vee \neg(x \in B)$ De Morgan

$\Leftrightarrow x \notin A \vee x \notin B$

$\Leftrightarrow x \in \bar{A} \vee x \in \bar{B}$ def'n of comp.

$\Leftrightarrow x \in \bar{A} \cup \bar{B}$ def'n of $\cup$

this proves $\overline{A \cap B} = \bar{A} \cup \bar{B}$

③ is similar: you try.

__Theorem__ For any sets A, B, C we have

① $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

② $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Pf: you try (use the logical distributive laws)

Proof writing

__Always two approaches:__ when trying to prove a statement P, can either prove P directly, __or__ assume $\neg P$ and derive a contradiction.

__More generally:__ can prove any statement logically equiv. to P, or disprove any statement logically equiv to $\neg P$.

__Existence Claims__

__General form:__ $(\exists x \in S) P(x)$