Prop'n 1 For any sets A, B, C
If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

Pf: - Sps $x \in A$ is a fixed, arbitrary element of A
- Since $A \subseteq B$, we have $x \in B$, by def'n of $\subseteq$
- Then, since $B \subseteq C$, we have $x \in C$ again by def'n of $\subseteq$.
- Since $x \in A$ was arbitrary, the same argument would apply to any el't of A.
- Hence every el't of A is an el't of C, i.e. $A \subseteq C$. ✓

More ex's: ④ For any set X, we have $X \subseteq X$
why: fix $x \in X$. Then $x \in X$ too...
⑤ set-builder notation defines a subset
i.e. if $Y = \{x \in X \mid x \text{ has } P\}$ then $Y \subseteq X$.
⑥ For any set X we have $\emptyset \subseteq X$.
↳ perhaps unintuitive, here's why:

Pf: it is true that

<u>if</u> (i) $x \in \emptyset$

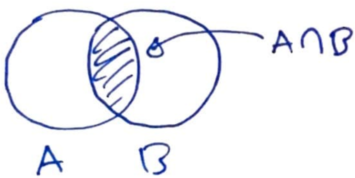
<u>then</u> (ii) $x \in X$ → more on this type of reasoning later

simply because (i) never holds!

Operations on Sets

<u>Intersections</u>: the intersection of two sets A, B – denoted $A \cap B$ – is the set of el'ts belonging to <u>both</u> A and B

i.e. $x \in A \cap B$ if and only if $x \in A$ <u>and</u> $x \in B$



<u>Ex</u>: if $A = \{1,2,3,4\}$, $B = \{1,3,5\}$, $C = \{2,4,6\}$ <u>then</u>: $A \cap B = \{1,3\}$, $A \cap C = \{2,4\}$, $B \cap C = \emptyset$

<u>Def'n</u> Two sets are called <u>disjoint</u> iff their intersection is $\emptyset$.

"if and only if"

e.g. B and C above are disjoint.

(2) **Prop'n**: For any sets A, B we have:

(i) $A \cap B \subseteq A$

(ii) $A \cap B \subseteq B$

↳ "obvious" from the picture, but let's prove from the def'ns for practice.

Pf: (i) – Fix $x \in A \cap B$

– then $x \in A$ and $x \in B$ by def'n of $\cap$

– hence in particular $x \in A$

– Since $x \in A \cap B$ was arbitrary, __every__ el't of $A \cap B$ is an el't of A, i.e. $A \cap B \subseteq A$.

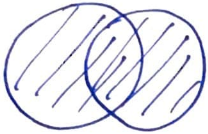
(ii) similar.

__Unions__: __Def'n__ the union of A and B, denoted $A \cup B$, is the set of el'ts contained in __either__ A __or__ B.

i.e. $x \in A \cup B$

iff $x \in A$ __or__ $x \in B$

Note: "or" here, as in all math, is non-exclusive.



$A \cup B$

A B

<u>Ex's</u> ① $\{1,3,5\} \cup \{2,4,6\} = \{1,2,3,4,5,6\}$
$= [6]$

② if $O = \{1,3,5,\ldots\}$ $E = \{2,4,6,\ldots\}$
then $O \cup E = \{1,2,3,4,5,\ldots\} = \mathbb{N}$

③ <u>Prop'n</u> For any sets A, B we have:
(i) $A \subseteq A \cup B$
(ii) $B \subseteq A \cup B$

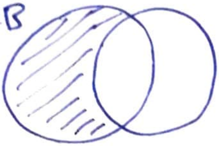
<u>Pf</u>: you try

<u>Difference</u>: <u>Def'n</u>: the <u>difference</u> of two sets A and B, denoted $A - B$, is the set of elements in A that are not in B

i.e. $x \in A - B$

iff $x \in A$ and $x \notin B$

A−B



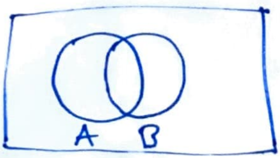
A B

<u>Ex</u>: if $A = \{1,2,3\}$ $B = \{3,4,5\}$
then $A - B = \{1,2\}$ and $B - A = \{4,5\}$

<u>notice</u>: difference is not a commutative operation, i.e. $A - B \neq B - A$ in general.
However $\cap$ and $\cup$ are commutative

i.e $A \cup B = B \cup A$
and $A \cap B = B \cap A$ always

<u>Note</u>: in defining $\cap, \cup, -$ it is sometimes convenient to assume our sets A, B are both subsets of a larger set U (called a <u>universal set</u>)



U

Then we can define these operations using set-builder notation:

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$

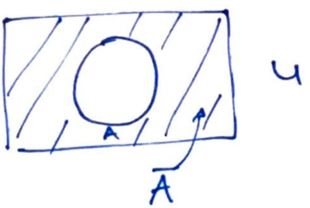
$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$

$$A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}$$

$$= \{x \in A \mid x \notin B\}$$

<u>Complement</u>: <u>Def'n</u> given a set A, and a universal set U with $A \subseteq U$, the <u>complement</u> of A, denoted $\bar{A}$, is the set of el'ts in U that are not in A

i.e. $\bar{A} = \{x \in U \mid x \notin A\}$



(note: really $\bar{A}$ is just $U - A$)

<u>Ex</u>: Sps $U = \mathbb{N}$

$A = \{1, 2, 3\}$

$E = \{2, 4, 6, \ldots\}$ $\quad O = \{1, 3, 5, \ldots\}$

then: $\bar{A} = \{4,5,6,\ldots\}$
$\bar{E} = \{1,3,5,\ldots\} = O$
$\bar{O} = \{2,4,6,\ldots\} = E$.
$\bar{N} = \emptyset$

Indexing by sets:

- often useful to take unions + intersections of more than 2 sets
  ↳ need notation for this!

Ex: For any $i \in \mathbb{N}$ define:

$$A_i = \{-i, 0, i\}$$

so, e.g. $A_1 = \{-1,0,1\}$, $A_2 = \{-2,0,2\}$, etc.

then: $A_1 \cup A_2 = \{-2,-1,0,1,2\}$
$A_1 \cup A_2 \cup A_3 = \{-3,-2,-1,0,1,2,3\}$

or even $A_1 \cup A_2 \cup \cdots \cup A_{10} = \{-10,-9,\ldots,0,1,\ldots,9,10\}$

↳ we could denote the above union as

$$\bigcup_{i=1}^{10} A_i$$

- but instead we'll think of the index variable i as ranging over <u>the set</u> $[10] = \{1, 2, \dots, 9, 10\}$ and write

$$\bigcup_{i \in [10]} A_i \quad \text{or, equivalently} \quad \bigcup_{i \in \{1,2,\dots,10\}} A_i$$

- in the same way we could write

$$\bigcup_{i \in \{1,2\}} A_i \quad \text{for} \quad A_1 \cup A_2$$

$$\bigcup_{i \in \{1,2,3\}} A_i \quad \text{for} \quad A_1 \cup A_2 \cup A_3$$

<u>More generally</u>:

<u>Def'n</u>: Sps that I is a set (called the <u>index set</u>) s.t. for every $i \in I$ we've defined a set A_i.

we define

$\bigcup_{i \in I} A_i$ as the set of el'ts contained in <u>at least one</u> of the A_i

i.e. $x \in \bigcup_{i \in I} A_i$ iff there exists $i \in I$ s.t. $x \in A_i$

We also define:

$\bigcap_{i \in I} A_i$ as the set of el'ts contained in <u>every</u> A_i

i.e. $x \in \bigcap_{i \in I} A_i$ iff, for <u>every</u> $i \in I$, we have $x \in A_i$.

<u>Ex's</u>: For $i \in \mathbb{N}$, define $A_i = \{-i, 0, i\}$ as before

① Let $I = [10] = \{1, 2, \dots, 10\}$

Then $\bigcup_{i \in I} A_i = \bigcup_{i \in [10]} A_i = \{-10, -9, \dots, 8, 9, 10\}$

whereas $\bigcap_{i \in I} A_i = \{0\}$

② An infinite union:

$$\bigcup_{i \in \mathbb{N}} A_i = \{\dots -2, -1, 0, 1, 2, 3, \dots\} = \mathbb{Z}$$

③ Let $E = \{2, 4, 6, \dots\}$

then $\bigcup_{i \in E} A_i = \{\dots -2, 0, 2, 4, \dots\}$

④ It may be that the indices themselves are sets!

e.g. let $X = \{\{1,2\}, \{1,3\}, \{1,4\}\}$

what is $\bigcup_{y \in X} y$

↪ the union of all sets in X!

$$\bigcup_{y \in X} y = \{1,2\} \cup \{1,3\} \cup \{1,4\}$$
$$= \{1,2,3,4\}$$

Similarly: $$\bigcap_{y \in X} y = \{1,2\} \cap \{1,3\} \cap \{1,4\}$$
$$= \{1\}$$

Indexing sets of sets: notation:

If we've defined A_i for every $i \in I$ we'll write $\{A_i : i \in I\}$ for the set of all sets A_i, for $i \in I$.

e.g. if $I = [3] = \{1,2,3\}$ and $A_i = \{-i, 0, i\}$ as above, then:

$\{A_i : i \in [3]\}$ denotes the set

$$\{A_1, A_2, A_3\} = \{\{-1,0,1\}, \{-2,0,2\}, \{-3,0,3\}\}$$

Equality of Sets

- a set is determined by its elements, two sets are equal exactly when they contain the same elements.

More precisely:

Def'n: For sets A, B we define

$A = B$ iff ① $A \subseteq B$

and ② $B \subseteq A$

i.e. $A = B$ iff whenever $a \in A$ then $a \in B$ and whenever $b \in B$ then $b \in A$

→ main import of def'n is in proofs; to prove $A = B$ one proves:

① $A \subseteq B$ and ② $B \subseteq A$

→ more on this later.