Induction

Q: What is the sum of the first n odd integers?

$1 = 1 = 1^2$

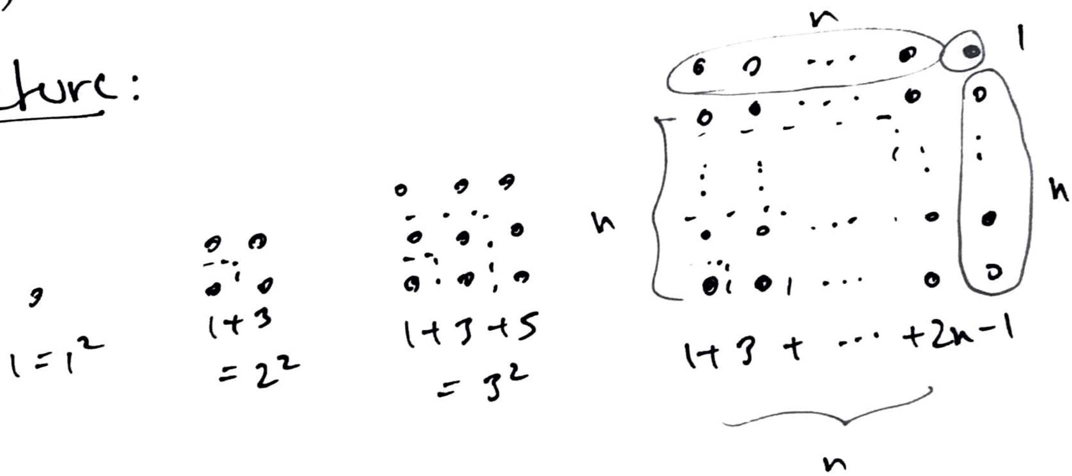
$1 + 3 = 4 = 2^2$

$1 + 3 + 5 = 9 = 3^2$

$1 + 3 + 5 + 7 = 16 = 4^2$

(*) $1 + 3 + 5 + \dots + (2n-1) \overset{??}{=} n^2$

Picture:



$n^2 + (2n+1)$

$= (n+1)^2$

- the picture suggests (*) is true for all $n \in \mathbb{N}$. How would we prove it? (75)

- pic also suggests the the proof for $n+1$ depends on proof for n.

Theorem For every $n \in \mathbb{N}$, we have:

$$1+3+5+\cdots+(2n-1) = n^2$$

i.e. $$\sum_{k=1}^{n} 2k-1 = n^2$$

Pf: - identity is clearly true when $n=1$ since $\sum_{k=1}^{1} 2k-1 = 1 = 1^2$.

- fix $n \in \mathbb{N}$ and suppose that we have the identity for n, that is, assume that

$$\sum_{k=1}^{n} 2k-1 = n^2$$

- now consider the sum for $n+1$:

$$\sum_{k=1}^{n+1} 2k-1 = \underbrace{1+3+\cdots+2n-1}_{\sum_{1}^{n} 2k-1} + 2(n+1)-1$$

$=$

$$= \sum_{k=1}^{n} 2k-1 \quad + 2n+1$$

$$= n^2 \quad + 2n+1 \qquad = (n+1)^2$$

by our assumption

we've shown:

(a) identity holds for $n=1$

(b) if it holds for a fixed $n \in \mathbb{N}$,
then it also holds for $n+1$

It follows: since identity holds for $n=1$
it also holds for $n=2$,
and so for $n=3$,
and so for $n=4$, ..
:
and so for all $n \in \mathbb{N}$!

The validity of this type of argument is called the principle of mathematical induction. (PMI)

"Theorem" (PMI) Suppose $P(n)$ is a variable prop'n. Suppose further that:

e.g. "$\sum_{k=1}^{n} 2k-1 = n^2$"

① $P(1)$ holds
② $(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n+1))$ holds

then: $(\forall n \in \mathbb{N})\, P(n)$ holds.

↳ for a "proof" see the book.
↳ we'll take PMI as an axiom
(i.e. we'll assume the type of reasoning we used above is valid)
↳ later, we'll show PMI is equivalent to another intuitively obvious principle.

Using PMI to prove $(\forall n \in \mathbb{N})\, P(n)$

① (Base case) Verify $P(1)$ directly
② (Induction hypothesis) Fix $n \in \mathbb{N}$. Assume $P(n)$ holds
③ (Induction step) using this hypothesis, deduce $P(n+1)$