(IS) Consider:

$$\sum_{k=1}^{n+1} f_k = \sum_{k=1}^{n} f_k + f_{n+1}$$

$$\overset{IH}{=} f_{n+2} - 1 + f_{n+1}$$

$$= f_{n+1} + f_{n+2} - 1$$

$$\overset{\text{def'n of Fib. seq.}}{=} f_{n+3} - 1$$

$$= f_{(n+1)+2} - 1$$

by PMI: $\forall n \in \mathbb{N}$ we have $\sum_{k=1}^{n} f_k = f_{n+2} - 1$ ✓

Prop'n: If n is a multiple of 3. (i.e. $n \in \{3, 6, 9, \ldots\}$) then f_n is even.

Pf (BC) if $n = 3$, then $f_n = 2$, which is even.

(IH) Fix $n \in \{3, 6, 9, \ldots\}$ and assume f_n is even

(IS) Consider f_{n+3}:

$$f_{n+3} = f_{n+2} + f_{n+1}$$

$= (f_{n+1} + f_n) + f_{n+1}$

$= 2f_{n+1} + f_n$

by the IH, f_n is even. Since $2f_{n+1}$ is even, $2f_{n+1} + f_n$ is as well, i.e. f_{n+3} is even. ✓

By induction, $\forall n \in \{3, 6, 9, \ldots\}$ f_n is even

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Strong Induction

- in certain proofs, may need to assume more than $P(n)$ to prove $P(n+1)$.
- e.g. may need to assume $P(n)$ and $P(n-1)$ ... or even $P(n), P(n-1), \ldots, P(1)$
- still a legit induction hypothesis!

"Thm" (Principle of strong mathematical induction) (PSMI)

- Sps $P(n)$ is a variable propn

If ① $P(1)$ holds (93)
② $(\forall n \in \mathbb{N}) [(\forall k \in [n]) P(k) \Rightarrow P(n+1)]$ holds

Then $(\forall n \in \mathbb{N}) P(n)$ holds

"$\forall k \leq n$"

Template for a PSMI proof:

① Prove $P(1)$
② Fix $n \in \mathbb{N}$. Assume $(\forall k \in [n]) P(k)$
(i.e. assume $P(1) \wedge P(2) \wedge \cdots \wedge P(n)$)
③ Deduce $P(n+1)$

→ PSMI then gives: $(\forall n \in \mathbb{N}) P(n)$ holds.

Note: - despite name, PSMI seems weaker than PMI, since we have to assume more (namely $P(1) \wedge P(2) \wedge \cdots \wedge P(n)$ instead of just $P(n)$) to prove $P(n+1)$

- but: we'll later show PMI and PSMI are equivalent (and both equiv. to another principle called WOP)

Ex: Let s_n be the sequence defined recursively by:

$$s_0 = 1$$
$$s_n = 1 + \sum_{k=0}^{n-1} s_k \quad \text{for } n \geq 1$$

so e.g. $s_1 = 1 + s_0 = 1 + 1 = 2$
$s_2 = 1 + s_0 + s_1 = 1 + 1 + 2 = 4$
$s_3 = 1 + s_0 + s_1 + s_2 = 1 + 1 + 2 + 4 = 8$

↳ Looks like $s_n = 2^n$

Let's prove - we'll need a strong inductive hypothesis.

Prop'n $\forall n \in \mathbb{N} \cup \{0\}$ we have $s_n = 2^n$

Pf: (BC) if $n = 0$ then $s_0 = 1 = 2^0$

(Strong IH) Fix $n \in \mathbb{N} \cup \{0\}$ and assume for every $k \in \{0, 1, \ldots, n\}$ we have $s_k = 2^k$

(IS) Now consider:

$$s_{n+1} = 1 + \sum_{k=0}^{n} s_k$$

$$= 1 + \sum_{k=0}^{n} 2^k$$

by strong IH

$$= 1 + \frac{2^{n+1}-1}{2-1}$$

by geometric series formula

Recall: if $x \neq 0, 1$, then $\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x-1}$

$$= 2^{n+1} \checkmark$$

$\Rightarrow$ by PSMI, $S_n = 2^n$ for every $n \in \mathbb{N} \cup \{0\}$.

Notice: we really needed a strong IH, since we had to replace every term in the sum $\sum S_k$ by 2^k, not just the nth term.