<u>Thm</u> The following are equivalent:

1. PMI
2. PSMI
3. WOP

i.e. PMI $\Leftrightarrow$ PSMI $\Leftrightarrow$ WOP

i.e. from any of these statements can prove the other two.

<u>Pf</u>: We've already shown:

PSMI $\Rightarrow$ WOP

Hence if we can show:

<u>Claim 1</u>: PMI $\Rightarrow$ PSMI

<u>Claim 2</u>: WOP $\Rightarrow$ PMI

$$\begin{tikzcd} & PMI \arrow[dr, Rightarrow, "\text{Claim 1}"] & \\ WOP \arrow[ur, Rightarrow, "\text{Claim 2}"] & & PSMI \arrow[ll, Rightarrow, "\text{know}"] \end{tikzcd}$$

we will have established the equivalence of all three statements.

Before proving PMI $\Rightarrow$ PSMI, let's illustrate idea of proof w/ an example.

Recall our first strong induction pf:

Define $s_0 = 1$

$$s_n = 1 + \sum_{k=0}^{n-1} s_k \quad \text{for } n \geq 1$$

<u>Claim</u> $\forall n \in \mathbb{N} \cup \{0\}$ we have $\underbrace{s_n = 2^n}_{P(n)}$

<u>Pf</u>: We proved this w/ PSMI

- To prove using just PMI we can "hack" a strong inductive hypothesis into the statement we induct on.
- Let $Q(n)$ be "$(\forall k \in \{0, \ldots n\})(s_k = 2^k)$"
- if we can prove $Q(n)$ holds for all n, then in particular we've proved $s_n = 2^n$ holds for all n, i.e. $P(n)$ holds for all n.

(BC) $Q(0)$ holds since this is just $(\forall k \in \{0\}) s_k = 2^k$ which is true since $s_0 = 1 = 2^0$.

(IH) Fix $n \in \mathbb{N} \cup \{0\}$ and assume $Q(n)$ (105)

i.e. assume $(\forall k \in \{0,1,\dots,n\})(s_k = 2^k)$

(IS) to prove $Q(n+1)$, i.e.

$(\forall k \in \{0,1,\dots,n,n+1\})(s_k = 2^k)$

It is sufficient to prove $s_{n+1} = 2^{n+1}$ since our IH already gives $\forall k \in \{0,1,\dots,n\}\ s_k = 2^k$

Observe:

$$\begin{aligned} s_{n+1} &\overset{\text{defn of seq}}{=} 1 + \sum_{k=0}^{n} s_k \\ &= 1 + \sum_{k=0}^{n} 2^k \quad \leftarrow \text{IH} \\ &= 1 + \frac{2^{n+1} - 1}{2 - 1} \quad \text{(geo series formula)} \\ &= 2^{n+1} \end{aligned}$$

So by regular induction we've proved $(\forall n \in \mathbb{N} \cup \{0\})\ Q(n)$ which as noted gives $(\forall n \in \mathbb{N} \cup \{0\})\ P(n)$ i.e. $(\forall n \in \mathbb{N})\ s_n = 2^n$.

Proof of PMI $\Rightarrow$ PSMI

Assume PMI: for every prop'n $P(n)$, if ① $P(1)$ and ② $(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n+1))$ hold, then $(\forall n \in \mathbb{N}) P(n)$ holds.

Want to prove: PSMI: for every prop'n $Q(n)$, if ⓐ $Q(1)$ and ⓑ $(\forall n \in \mathbb{N})[(\forall k \in [n]) Q(k) \Rightarrow Q(k+1)]$ hold, then $(\forall n \in \mathbb{N}) Q(n)$ holds.

- so fix a prop'n $Q(n)$ and assume ⓐ and ⓑ hold.
- let $P(n)$ be the prop'n
  "$(\forall k \in [n]) Q(k)$" $\hookrightarrow$ we'll prove $(\forall n \in \mathbb{N}) P(n)$ by PMI

(BC) $P(1)$ holds, since this is $(\forall k \in [1]) Q(k)$, which is equiv. to $Q(1)$, which holds by assumption ⓐ.

(IH) Fix $n \in \mathbb{N}$ and assume $P(n)$, i.e. assume $(\forall k \in [n]) Q(k)$ holds.

(IS) Then by ⓑ we know $Q(n+1)$ holds; hence $(\forall k \in [n]) Q(k)$ and $Q(n+1)$ hold, hence $(\forall k \in [n+1]) Q(k)$ holds, i.e. $P(n+1)$ holds

by (regular) induction (PMI)
we have $(\forall n \in \mathbb{N})\, P(n)$ holds, i.e.
$(\forall n \in \mathbb{N})(\forall k \in [n])\, Q(k)$ holds. But this
implies $(\forall n \in \mathbb{N})\, Q(n)$ holds, as desired. ✓

<u>Proof of WOP $\Rightarrow$ PMI</u>: Assume WOP.

We want to prove PMI. So fix a
prop'n $P(n)$ and assume

① $P(1)$
② $(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n+1))$ both hold.

We prove $(\forall n \in \mathbb{N})\, P(n)$ holds.

<u>idea</u>: Let $S = \{n \in \mathbb{N} \mid P(n) \text{ fails}\}$
we'll use WOP to prove $S = \emptyset$.

- if $S \neq \emptyset$, then by WOP, S has a least element x.
- we know $x \neq 1$, since $P(1)$ holds.
- hence $x = n+1$ for some $n \in \mathbb{N}$.
- since x is <u>least</u> number for which P is false, must have that $P(n)$ holds.

- but then by (2), $P(n+1)$ holds, i.e. $P(x)$ holds, a contradiction, as $P(x)$ fails.
- Hence $S = \emptyset$, which gives $(\forall n \in \mathbb{N}) P(n)$ holds, as desired.

↳ PMI, PSMI, WOP are all intuitively obvious principles, and often taken as axioms.

↳ we've proved: if you assume any one of them, can prove other two.

Binary relations

- binary relations are ubiquitous in math.
- e.g. we have order relations like:

$$x \leq y$$
$$x < y$$

- the subset relation

$$X \subseteq Y$$

- the divisibility relation

$n \mid m$ ("n divides m")

↳ all assert a relation between two objects (hence "binary")

↳ but what are $<, \leq, \subseteq, \mid$ as math'l objects themselves?

↳ we will define binary relations as sets of ordered pairs

Def'n Sps A, B are sets. A binary relation on A and B is simply a subset

$$R \subseteq A \times B.$$

→ if $(a,b) \in R$ we say "a is related to b" and sometimes write aRb.

↳ we say A is the <u>domain</u> of R, B is the <u>codomain</u>.

↳ often we have $A = B$ so that $R \subseteq A \times A$. In this case we say: R is a <u>relation</u> <u>on</u> <u>A</u>.

<u>Ex's</u> ① Let A = set of Shakespeare's characters

B = set of Shakespeare's plays.

- Define a relation $R \subseteq A \times B$ by:

$(a,b) \in R$ iff a appears in b.

i.e. $R = \{(a,b) \in A \times B \mid a \text{ appears in } b\}$

- <u>then</u>: (Romeo, "Romeo and Juliet") $\in R$

(Iago, "Othello") $\in R$

(Romeo, "Othello") $\notin R$.

- might also write: Romeo R "Romeo and Juliet"

Romeo $\not R$ "Othello"

② Consider $\leq$ and $<$ as relations on $\mathbb{N}$. Can think of them as sets of pairs.