$\leq \; = \{(a,b) \in \mathbb{N}\times\mathbb{N} \mid a \text{ is not greater than } b\}$

$= \{(1,1), (1,2), (5,217), \dots\}$

$< \; = \{(a,b) \in \mathbb{N}\times\mathbb{N} \mid a \text{ is strictly less than } b\}$

$= \{(1,2), (17,200), \dots\}$

- we write $1 \leq 2$ instead of $(1,2) \in \leq$, but these mean the same thing.
- likewise $2 \not< 1$ means $(2,1) \notin <$.
- can visualize $\leq$ and $<$ as "triangular" subsets of the grid $\mathbb{N}\times\mathbb{N}$:

$$\begin{array}{ccccc}
(1,4) & (2,4) & (3,4) & (4,4) & \dots \\
(1,3) & (2,3) & (3,3) & (4,3) & \dots \\
(1,2) & (2,2) & (3,2) & (4,2) & \dots \\
(1,1) & (2,1) & (3,1) & (4,1) & \dots
\end{array}$$

$<$

$\leq$

③ Sps A is a set. Can think of $=$ as a binary relation on A

$=$ is the set $\{(x,y) \in A\times A \mid x = y\}$

i.e. $\{(x,x) : x \in A\}$

④ relations are arbitrary sets of pairs and need not be defined by some intelligible property, e.g. (112)

- $R = \{(1,1), (2,\pi), (3,\sqrt{2})\} \subseteq \mathbb{R}\times\mathbb{R}$ is a relation on $\mathbb{R}$.

- $\emptyset$ is a relation on <u>every</u> set A ("the empty relation")

<u>Properties relations can have:</u>

<u>Def'n</u> Spr A is a set and $R \subseteq A\times A$ is a relation on A.

① R is <u>reflexive</u> iff
$(\forall x \in A)\,((x,x)\in R)$

② R is <u>symmetric</u> iff
$(\forall x,y \in A)\,((x,y)\in R \Rightarrow (y,x)\in R)$

③ R is <u>transitive</u> iff
$(\forall x,y,z \in A)\,(((x,y)\in R \wedge (y,z)\in R) \Rightarrow (x,z)\in R)$

④ R is <u>antisymmetric</u> iff
$(\forall x,y \in A)\,(((x,y)\in R \wedge (y,x)\in R) \Rightarrow x=y)$

Ex's ① on any set A, the equality relation $=$ is reflexive, symmetric, and transitive (and also antisymmetric, trivially)

↳ relations w/ these three properties are called equivalence relations (more later.)

② $\leq$ (e.g. on $\mathbb{N}$) is reflexive, transitive, and antisymmetric:

why: $(\forall n \in \mathbb{N})(n \leq n)$

$(\forall n, m, l \in \mathbb{N})(n \leq m \wedge m \leq l \Rightarrow n \leq l)$

$(\forall n, m \in \mathbb{N})(n \leq m \wedge m \leq n \Rightarrow n = m)$

but $\leq$ is not symmetric since eg.

$3 \leq 5$ but $5 \not\leq 3$.

③ $<$ (e.g. on $\mathbb{N}$) is not reflexive or symmetric, but is transitive

(is $<$ antisymmetric? Why?)

④ Let $A = \{$rock, paper, scissors$\}$

define a relation R on A by:

$(a, b) \in R$ iff a beats b.

is R transitive? Nope:

(scissors, paper) $\in R$

(paper, rock) $\in R$

but (scissors, rock) $\notin R$

5 Consider the divisibility relation $|$ on $\mathbb{N}$:

$n | m$ iff n divides m (i.e. $(\exists k \in \mathbb{N})(m = nk)$)

e.g. $2|4$ and $2|12$ but $2 \nmid 7$

<u>Claim</u>: divisibility on $\mathbb{N}$ is:

1. reflexive
2. <u>not</u> symmetric
3. transitive
4. antisymmetric

<u>Pf</u>: 1 Fix $n \in \mathbb{N}$. Then $n|n$ since $n = n \cdot 1$.

2 e.g. $2|4$ but $4 \nmid 2$.

3 Fix $n, m, \ell \in \mathbb{N}$. Sps $n|m$ and $m|\ell$.

i.e. $\exists k_1, k_2 \in \mathbb{N}$ s.t. $m = nk_1$
$\ell = mk_2$

but then $\ell = mk_2 = (nk_1)k_2 = n(k_1k_2)$

hence $n|\ell$

4 Fix $n, m \in \mathbb{N}$ and sps $n \mid m$ and $m \mid n$.

Then: $n = k_1 m$
$m = k_2 n$ for some $k_1, k_2 \in \mathbb{N}$

So: $n = k_1 k_2 n \Rightarrow k_1 k_2 = 1$
$\Rightarrow k_1 = k_2 = 1$
$\Rightarrow n = m$ ✓

5 Now consider divisibility on $\mathbb{Z}$:
$n \mid m$ iff $(\exists k \in \mathbb{Z})\ m = nk$

Then: $\mid$ remains reflexive, transitive.
But antisymmetric? No: e.g.
$2 \mid -2$ and $-2 \mid 2$
but $2 \neq -2$.

<u>Equivalence Relations</u>: <u>Def'n</u> a relation R on a set A is called an <u>equivalence relation</u> iff R is reflexive, symmetric, and transitive

<u>Ex's</u> 1 on any set A, the equality relation on A:
is an equivalence relation

<u>PF</u>: ~~[illegible]~~ $\forall x, y, z \in A$ we have:
(i) $x = x$ (ii) if $x = y$ then $y = x$ (iii) if $x = y$ and $y = z$ then $x = z$

Recall: the **floor** of a real number x, denoted $\lfloor x \rfloor$, is the unique integer n s.t. $n \leq x < n+1$. (116)

e.g.
$$\lfloor 1.5 \rfloor = 1$$
$$\lfloor \pi \rfloor = 3$$
$$\lfloor -2.07 \rfloor = -3$$
$$\lfloor 5 \rfloor = 5$$

Define a relation R on $\mathbb{R}$ by:

$(x,y) \in R$ iff $\lfloor x \rfloor = \lfloor y \rfloor$

i.e. $R = \{(x,y) \in \mathbb{R}^2 \mid \lfloor x \rfloor = \lfloor y \rfloor\}$

---

e.g. $(1.5, 1.25) \in R$ but $(1.5, 2.5) \notin R$.

Claim: R is an equivalence relation.

Pf: (i) Fix $x \in \mathbb{R}$. Then $\lfloor x \rfloor = \lfloor x \rfloor$. Hence $(x,x) \in R$. Since x was arbitrary, we have $(\forall x \in \mathbb{R})((x,x) \in R)$

(ii) Fix $x, y \in \mathbb{R}$ and sps $(x,y) \in R$. i.e. $\lfloor x \rfloor = \lfloor y \rfloor$. Hence $\lfloor y \rfloor = \lfloor x \rfloor$ (by the symmetry of equality), so $(y,x) \in R$ ✓

(iii) Fix $x,y,z \in \mathbb{R}$. Sps $(x,y),(y,z) \in R$ so that $\lfloor x \rfloor = \lfloor y \rfloor$ and $\lfloor y \rfloor = \lfloor z \rfloor$. Hence $\lfloor x \rfloor = \lfloor z \rfloor$. Hence $(x,z) \in R$.

(3) Define a relation $\equiv_3$ on $\mathbb{Z}$ by:

$(n,m) \in \equiv_3$ iff $3 \mid (m-n)$ ← $\exists k \in \mathbb{Z}$ $m-n = 3k$

i.e. $\equiv_3 = \{(n,m) \in \mathbb{Z}^2 \mid 3 \mid (m-n)\}$

↳ we'll usually write $n \equiv_3 m$ instead of $(n,m) \in \equiv_3$.

e.g. $2 \equiv_3 5$ since $3 \mid (5-2)$

$7 \equiv_3 -2$ since $3 \mid (-2-7)$

$6 \not\equiv_3 7$ since $3 \nmid (7-6)$

Claim: $\equiv_3$ is an equiv. relation on $\mathbb{Z}$.

Pf: (i) Fix $n \in \mathbb{Z}$. Observe $3 \mid n-n$, i.e. $3 \mid 0$ since $0 = 3 \cdot 0$. Hence $n \equiv_3 n$.

(ii) Fix $n,m \in \mathbb{Z}$ and sps $n \equiv_3 m$ i.e. $3 \mid m-n$. We prove $m \equiv_3 n$.

$\exists k \in \mathbb{Z}$ s.t. $m-n = 3k$

but then

$$n-m = 3(-k)$$

hence $3 \mid n-m$, so $m \equiv_3 n$ ✓

(iii) Fix $n, m, l \in \mathbb{Z}$ and sps $n \equiv_3 m$ and $m \equiv_3 l$. We prove $n \equiv_3 l$.

we know: $\exists k_1, k_2 \in \mathbb{Z}$ s.t.

$$m-n = 3k_1$$
$$l-m = 3k_2$$

adding these equations gives:

$$(m-n)+(l-m) = 3k_1 + 3k_2$$

$\Rightarrow$ ~~[illegible]~~ $l-n = 3(k_1+k_2)$

$\Rightarrow$ $3 \mid l-n$

Hence $n \equiv_3 l$. ✓

↳ $\equiv_3$ is called congruence modulo 3.

↳ it's more conventional to write $n \equiv m \pmod 3$ instead of $n \equiv_3 m$

(we'll use these notations interchangeably)

Another way to think about it:

$n \equiv m \pmod 3$ iff n, m have the same remainder when divided by 3.

e.g. $2 \equiv 5 \pmod 3$

since $2 = 3 \cdot 0 + 2$
$5 = 3 \cdot 1 + 2$ same remainder

$7 \equiv 13 \pmod 3$

since $7 = 3 \cdot 2 + 1$
$13 = 3 \cdot 4 + 1$ same

$7 \equiv -2 \pmod 3$

since $7 = 3 \cdot 2 + 1$
$-2 = 3(-1) + 1$

but $7 \not\equiv 11 \pmod 3$

since $7 = 3 \cdot 2 + 1$
$11 = 3 \cdot 3 + 2$ different

(4) Nothing special about $n = 3$. For any fixed $n \in \mathbb{N}$ can define $\equiv_n$ on $\mathbb{Z}$ by:

$$a \equiv_n b \text{ iff } n \mid (b - a)$$

- just like w/ 3 we'll usually write $a \equiv b \pmod{n}$ for $a \equiv_n b$.
- just like with 3, $\equiv_n$ is an equiv. relation for any fixed $n \in \mathbb{N}$.