__Notice__: in both ex's ① and ② the set of equiv. classes forms a partition of the underlying set ($\mathbb{Z}$ in ①, $\mathbb{R}$ in ②)

– turns out this is always the case:

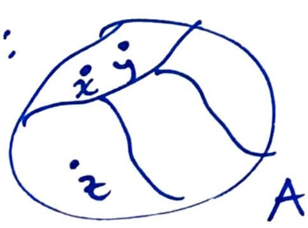
__Theorem__ If R is an equiv. relation on A, then A/R is a partition of A.

Pf: (HW) For a hint, see 6.7.13 on pg. 449, or go to recitation tomorrow.

Partitions yield equiv. relations

__idea__: if $\mathbb{P}$ is a partition of A, can define an equiv. relation R on A by "$(x,y) \in R$ iff x and y are in the same piece of the partition."

__Pic__:



$(x,y) \in R$

$(x,z) \notin R$

Let's prove this works:

<u>Thm</u> Sps $\mathbb{P}$ is a partition of A.
Define a relation $R_{\mathbb{P}}$ by:

$$(x,y) \in R_{\mathbb{P}} \text{ iff } (\exists X \in \mathbb{P})(x \in X \wedge y \in X)$$

<u>Then</u>: $R_{\mathbb{P}}$ is an equiv. relation.

<u>Pf</u>: (i) (reflexivity)

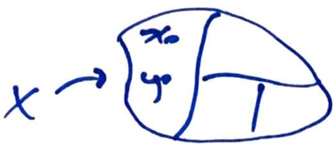
- Fix $x \in A$
- Since $\mathbb{P}$ is a partition we have

$$\bigcup_{X \in \mathbb{P}} X = A$$

- hence $x \in \bigcup_{X \in \mathbb{P}} X$, so $\exists X \in \mathbb{P}$ s.t. $x \in X$.
- hence $(x,x) \in R_{\mathbb{P}}$

(ii) (symmetry)

- Fix $x, y \in A$ and suppose $(x,y) \in R_{\mathbb{P}}$
- then $\exists X \in \mathbb{P}$ s.t $x \in X \wedge y \in X$
- but then $y \in X \wedge x \in X$
- so $(y,x) \in R_{\mathbb{P}}$



(iii) (transitivity)

- Fix $x, y, z \in A$ and sps $(x,y) \in R_{\mathbb{P}}$ and $(y,z) \in R_{\mathbb{P}}$.
- then $\exists X \in \mathbb{P}$ s.t. $x \in X \wedge y \in X$
  and $\exists Y \in \mathbb{P}$ s.t. $y \in Y \wedge z \in Y$
- hence $y \in X \cap Y$, so $X \cap Y \neq \emptyset$.
- but then it must be $X = Y$ since $\mathbb{P}$ is a partition
- hence $x \in X$ and $z \in \overset{\text{"}Y}{X}$.
- hence $(x,z) \in R_{\mathbb{P}}$. ✓

Ex's: let $\mathbb{P} = \{X, Y, Z\}$ where

$$X = \{\ldots -3, 0, 3, 6, \ldots\}$$
$$Y = \{\ldots -2, 1, 4, 7, \ldots\}$$
$$Z = \{\ldots -1, 2, 5, 8, \ldots\}$$

be our partition of $\mathbb{Z}$ from before, and let $R_{\mathbb{P}}$ be the associated equiv relation: $(x,y) \in R_{\mathbb{P}}$ iff $(\exists S \in \{X, Y, Z\})((x,y) \in S)$

- by our theorem $R_{\mathbb{P}}$ really is an equiv. relation easy to see: this is the same equiv. relation as $\equiv_3$ which we defined before in a different way:

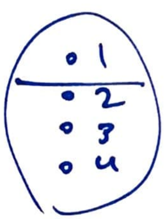
$$x \equiv_3 y \quad \text{iff} \quad 3 \mid y-x$$

notice: the equiv. classes of $R_{\mathbb{P}}$ are exactly the pieces of the partition. This always happens, i.e.

$$A/R_{\mathbb{P}} = \mathbb{P}.$$

② - Let $\mathbb{P} = \{\{1\}, \{2,3,4\}\}$ be our partition of $A = \{1,2,3,4\}$.

- Let $R_{\mathbb{P}}$ be the assoc. equiv. relation.



e.g. $(1,1) \in R_{\mathbb{P}}$

$(2,3) \in R_{\mathbb{P}}$

but $(1,2) \notin R_{\mathbb{P}}$

- we can actually write out $R_{\mathbb{P}}$ as a set of ordered pairs:

$$R_{\mathbb{P}} = \{(1,1), (2,2), (3,3)\ (4,4), (2,3)\ (2,4)\ (3,4)\ (3,2)\ (4,2)\ (4,3)\}$$

- no apparent rhyme or reason to this equiv. relation — but still a perfectly good one.

Order relations

- another common type of binary relation is an order relation
- unlike equivalence relations, order relations come in several flavors:

nonstrict / strict
partial / total

Def'n a relation R on a set A is a (nonstrict) partial order iff R is reflexive, transitive, and antisymmetric

↳ if R is a partial order on A, we say the pair (A, R) is a partially ordered set (or poset).

Ex's ① $\leq$ is a partial order on $\mathbb{R}$.

why: $\forall x, y, z \in \mathbb{R}$ we have:

(i) $x \leq x$ (ii) $x \leq y \wedge y \leq z \Rightarrow x \leq z$
(iii) $x \leq y \wedge y \leq x \Rightarrow x = y$

- so we'd say $(\mathbb{R}, \leq)$ is a poset

2. Let A be a fixed set. Consider the subset relation $\subseteq$ on $P(A)$. This is a partial order.

<u>Why</u>: $\forall X, Y, Z \in P(A)$ we have:

(i) $X \subseteq X$ ✓

(ii) $X \subseteq Y \wedge Y \subseteq Z \Rightarrow X \subseteq Z$ ✓

(iii) $X \subseteq Y \wedge Y \subseteq X \Rightarrow X = Y$ ✓

3. We showed that the divisibility relation $\mid$ on $\mathbb{N}$ is reflexive, transitive, and antisymmetric. Hence $\mid$ is a partial order and $(\mathbb{N}, \mid)$ is a poset.

Is $(\mathbb{Z}, \mid)$ a poset?

No! We observed that $\mid$ is no longer antisymmetric on $\mathbb{Z}$:

e.g. $2 \mid -2$ and $-2 \mid 2$

but $2 \neq -2$.

Hence $(\mathbb{Z}, \mid)$ is <u>not</u> a poset

These examples of partial orders seem to be of different kinds. And yet – any theorem that can be proved on the basis of only reflexivity, transitivity, and antisymmetry will be true for <u>all</u> posets!

<u>Strict Partial Orders</u>

<u>Def'n</u> a relation R on A is <u>irreflexive</u> iff $(\forall x \in A)((x,x) \notin R)$

e.g. $\neq$ and $<$ are irreflexive since we <u>never</u> have $x \neq x$ or $x < x$

<u>Def'n</u> a relation R on a set A is called a <u>strict partial order</u> iff R is (i) irreflexive, (ii) transitive (iii) antisymmetric

<u>Ex's</u>: ① $<$ is a strict partial order on $\mathbb{R}$.

<u>why</u>: $\forall x, y, z \in \mathbb{R}$ we have: