These examples of partial orders seem to be of different kinds. And yet — any theorem that can be proved on the basis of only reflexivity, transitivity, and antisymmetry will be true for all posets!

Strict Partial orders

Def'n a relation R on A is irreflexive iff $(\forall x \in A)((x,x) \notin R)$

e.g. $\neq$ and $<$ are irreflexive since we never have $x \neq x$ or $x < x$

Def'n a relation R on a set A is called a strict partial order iff R is (i) irreflexive, (ii) transitive (iii) antisymmetric

Ex's: ① $<$ is a strict partial order on $\mathbb{R}$.

why: $\forall x, y, z \in \mathbb{R}$ we have:

(i) $x \not< x$

(ii) $x<y \wedge y<z \Rightarrow x<z$

(iii) $(x<y \wedge y<x) \Rightarrow x=y$

vacuously, since this is always false

as we can see from this example, combination of irreflexivity and antisymmetry is a bit strange.

<u>Def'n</u> A relation R on A is called <u>asymmetric</u> iff $(\forall x,y \in A)((x,y)\in R \Rightarrow (y,x)\notin R)$

e.g. $<$ is asymmetric since
$x<y \Rightarrow y \not< x$.

On HW you showed: R is asymmetric iff R is both irreflexive and antisymmetric

↳ <u>upshot</u>: could as well define a strict p.o. as a relation R that is (i) transitive and (ii) asymmetric.

↳ verifying these two conditions often simpler in practice.

e.g. to see $<$ is a strict p.o. on $\mathbb{R}$ can just observe: $\forall x, y, z \in \mathbb{R}$:

(i) $x<y \wedge y<z \Rightarrow x<z$
(ii) $x<y \Rightarrow y \not< x$

② Let A be a fixed set. Then $\subsetneq$ is a strict p.o. on $P(A)$.

Pf: $\forall X, Y, Z \in P(A)$

(i) $X \subsetneq Y \wedge Y \subsetneq Z \Rightarrow X \subsetneq Z$ (why?)
(ii) $X \subsetneq Y \Rightarrow Y \not\subsetneq X$ (why?)

nonexamples:

① $\leq$ is not a strict p.o. on $\mathbb{R}$ since $\leq$ is not irreflexive (in fact, $\leq$ is reflexive, which is stronger than not being irreflexive)

② OTOH $<$ is not a (nonstrict) p.o. on $\mathbb{R}$, since $<$ is not reflexive (in fact, $<$ is irreflexive)

③ More generally: a relation R cannot be both a strict and non-strict p.o. (unless $R = \emptyset$ is the empty relation)

4. The relation $\neq$ of inequality (e.g. on $\mathbb{N}$) is neither a strict nor non-strict p.o. since $\neq$ is not transitive

Total orders

Def'n: A relation R on A is called *total* iff

$$(\forall x, y \in A)\big((x,y) \in R \vee (y,x) \in R \vee x = y\big)$$

Def'n: (i) if R is a partial order on A that is also total, then R is called a *total order* on A.

(ii) if R is a strict p.o. on A that is also total, R is called a *strict total order* on A.

Ex's 1. $\leq$ is a total order on $\mathbb{R}$:

we know $\leq$ is a p.o. and further

$$(\forall x, y \in \mathbb{R})(x \leq y \vee y \leq x \vee x = y)$$

2. $\subseteq$ is not a total order on $P(\mathbb{N})$ since it is not total, e.g.:

if $X = \{1,2\}$ $\quad Y = \{2,3\}$

then $X \not\subseteq Y$ and $Y \not\subseteq X$ and $X \neq Y$.

③ Similarly, $|$ is a p.o. on $\mathbb{N}$ but not a total order since e.g. $3 \nmid 5$ and $5 \nmid 3$ and $3 \neq 5$.

④ $<$ is a strict total order on $\mathbb{R}$:

we knew it's a strict p.o. and it's total since:

$$(\forall x, y \in \mathbb{R})\,(x<y \vee y<x \vee x=y)$$

⑤ $\subsetneq$ is not a strict total order on $P(\mathbb{N})$: again it's not total.

Functions

- functions, like relations, are ubiquitous in math. But what "are" functions?
- intuitively: a function is a rule that assigns to each input x in a domain A a unique output in a codomain B.
- idea: can define functions rigorously as a special type of relation.

Def'n: a function (with domain A and codomain B) is a relation $f \subseteq A \times B$

s.t.: for every $a \in A$
there is a unique $b \in B$
s.t. $(a,b) \in f$

i.e. $(\forall a \in A)(\exists b \in B)\Big[(a,b) \in f \wedge$
$(\forall c \in B)\big((a,c) \in f \Rightarrow c = b\big)\Big]$

existence of an output for input a
uniqueness of output for a.

Notation: - we write $f : A \to B$ to mean that a relation $f \subseteq A \times B$ is a function

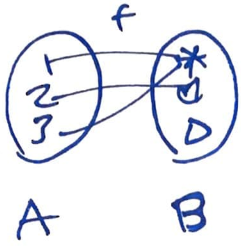
- we also write:
$f(a) = b$
to mean $(a,b) \in f$.

Note: - def'n says: every $a \in A$ assigned an output $b \in B$
- does not insist every $b \in B$ is mapped to by some input $a \in A$ (functions w/ this property are called surjective)

__Ex's__ Let $A = \{1,2,3\}$, $B = \{*, ♡, \Delta\}$

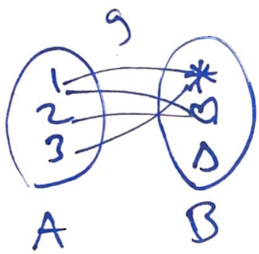
Then: $f = \{(1,*), (2,♡), (3,*)\}$

is a function from A to B:



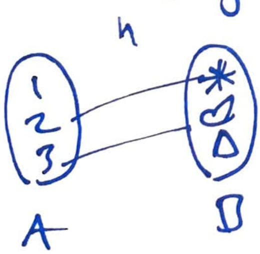
a function

but: $g = \{(1,*), (1,♡), (2,♡), (3,*)\}$ is __not__ a function since 1 does not have a unique output: both $(1,*), (1,♡) \in g$.



not a function

nor is $h = \{(2,*), (3,♡)\}$ since 1 is not assigned an output



not a function

(both g,h perfectly good relations)