PMI says: if you can do 1, 2, 3 then $(\forall n \in \mathbb{N}) P(n)$ holds. 78

Ex: 1 Q: what is the sum of the first n natural numbers?

$$1+2+\cdots+n = ?$$

first few:

$$\begin{aligned} &1 &&= 1 &&= 1\cdot 2/2 \\ &1+2 &&= 3 &&= 2\cdot 3/2 \\ &1+2+3 &&= 6 &&= 3\cdot 4/2 \\ &1+2+3+4 &&= 10 &&= 4\cdot 5/2 \end{aligned}$$

$$\hookrightarrow \quad 1+2+\cdots+n \overset{??}{=} \frac{n(n+1)}{2}.$$

Theorem For every $n \in \mathbb{N}$, we have:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Pf: Let $P(n)$ be the prop'n

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(BC): $P(1)$ is true since

$$\sum_{k=1}^{1} k = 1 = \frac{1\cdot 2}{2} \checkmark$$

(IH) Fix $n \in \mathbb{N}$ and assume $P(n)$,

i.e. assume $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

(IS) Now consider:

$$\sum_{k=1}^{n+1} k = \underbrace{1+2+\dots+n}_{\sum_{k=1}^{n} k} + n+1$$

$$= \sum_{k=1}^{n} k + (n+1)$$

by IH

$$= \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$= \frac{n(n+1) + 2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

$$= \frac{(n+1)((n+1)+1)}{2}$$

hence $P(n+1)$ holds.

by PMI, $P(n)$ holds for every $n \in \mathbb{N}$

i.e. $\forall n \in \mathbb{N}$ we have $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ ✓

<u>Notice</u>: proof doesn't really give insight into how we might have guessed the formula

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

But: once we have guessed, PMI gives us a way of verifying that it's really true $\forall n \in \mathbb{N}$.

② (Geometric series) Fix $x \in \mathbb{R}$ w/ $x \neq 0, 1$. Then for every $n \in \mathbb{N}$ we have:

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

i.e. $$\underbrace{\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}}_{\text{call this } P(n)}$$

(BC) $P(1)$ holds since

$$\sum_{k=0}^{0} x^k = x^0 \underset{\text{since } x \neq 0}{=} 1 \underset{\text{since } x \neq 1}{=} \frac{x^1 - 1}{x - 1}$$

(IH) Fix $n \in \mathbb{N}$ and assume $P(n)$

i.e. assume that $\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}$

(IS) Now consider:

$$\sum_{k=0}^{n} x^k = \sum_{k=0}^{n-1} x^k + x^n$$

$$\text{by IH} \quad = \frac{x^n - 1}{x - 1} + x^n$$

$$= \frac{x^n - 1}{x - 1} + \frac{x^n(x-1)}{x-1}$$

$$= \frac{x^n - 1 + x^{n+1} - x^n}{x - 1}$$

$$= \frac{x^{n+1} - 1}{x - 1}$$

$\Rightarrow P(n+1)$ holds

By PMI, $P(n)$ holds for all $n \in \mathbb{N}$

i.e. $\forall n \in \mathbb{N}$ we have:

$$\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}$$

<u>Prop'n</u> For every $n \in \mathbb{N}$, $7^n - 4^n$ is a multiple of 3.

<u>Pf (BC)</u> if $n=1$, the statement holds since $7^1 - 4^1 = 3$

(IH) Fix $n \in \mathbb{N}$ and assume $\exists k \in \mathbb{N}$ s.t. $7^n - 4^n = 3k$

(IS) Now, observe:

$$7^n = 3k + 4^n \quad \text{(by IH)}$$

$$\begin{aligned}
\Rightarrow 7^{n+1} &= (3k + 4^n)7 \\
&= 21k + 7 \cdot 4^n \\
&= 21k + (3+4) \cdot 4^n \\
&= 21k + 3 \cdot 4^n + 4^{n+1}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow 7^{n+1} - 4^{n+1} &= 21k + 3 \cdot 4^n = 3(7k + 4^n) \\
&= 3M
\end{aligned}$$

where $M = 7k + 4^n$

Hence $7^{n+1} - 4^{n+1}$ is a multiple of 3

By PMI, $7^n - 4^n$ is a multiple of 3 $\forall n \in \mathbb{N}$

Variants of induction

Nothing special about $n=1$ as a base case.

Thm (PMI w/ different BC)

- Sps $P(n)$ is a var. prop'n and $n_0 \in \mathbb{Z}$, possibly 0 or negative
- Let $S = \{n_0, n_0+1, n_0+2, \ldots\} = \{n \in \mathbb{Z} \mid n \geq n_0\}$

Then if we have

① $P(n_0)$ holds
② $(\forall n \in S)(P(n) \Rightarrow P(n+1))$ holds

then $(\forall n \in S) P(n)$ holds.

↳ can prove theorem using regular PMI (see book)

↳ proof template nearly the same as w/ PMI:

① (BC) verify $P(n_0)$
② (IH) Fix $n \in S$. Assume $P(n)$
③ (IS) Prove $P(n+1)$