(2) We'll often define functions by ~~[illegible]~~ a <u>rule</u>:

e.g. define $f: \mathbb{R} \to \mathbb{R}$
by $f(x) = x^2$
or: $g: \mathbb{R} \to \mathbb{R}$
$g(x) = \lfloor x \rfloor$

but behind the scenes these functions are still sets of ordered pairs:

$$f = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid y = x^2\}$$
$$g = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid y = \lfloor x \rfloor\}$$

so e.g. $(2,4) \in f$ but $(7,12) \notin f$
$(3,9) \in f$ since $f(7) = 49 \neq 12$.

<u>Warning</u>: not all rules yield well-defined functions!

e.g. suppose we "define"
$f: \mathbb{Q} \to \mathbb{Z}$
by the rule $f\left(\frac{m}{n}\right) = m+n$.

Then this "function" isn't one:

$f(\frac{1}{2}) = 1+2 = 3 \neq 6 = 2+4 = f(\frac{2}{4})$

but $\frac{1}{2} = \frac{2}{4}$.

- so f assigns multiple outputs to the same input. What's going on?
- really: there's an implicit <u>equiv relation</u> on fraction representations ($\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \ldots$) and our f is defined in terms of the representative of an equiv. class – not the class itself.
- in general: when given a rule "defining" $f \subseteq A \times B$, to verify f is a function one must show ① $(\forall a \in A)(\exists b \in B)[(a,b) \in f]$
② if $a = a'$ then $f(a) = f(a')$

<u>Equality of functions</u>

Q: What does it mean for two functions $f: A \to B$ and $g: A \to B$ to be equal?

A: $f = g$ iff they're equal as sets (of ordered pairs), i.e. $f \subseteq g$ and $g \subseteq f$.

In practice often easier to use following criterion:

<u>Prop'n</u>: if $f: A \to B$ and $g: A \to B$ are functions then $f = g$ iff

$$(\forall a \in A)\,(f(a) = g(a))$$

Pf: you try

<u>The point</u>: functions can be defined by different rules yet be equal.

E.g. let $A = \{1, 2, 3\}$

define $f: A \to \mathbb{N}$

$g: A \to \mathbb{N}$

by $f(x) = x^3 + 11x$

$g(x) = 6x^2 + 6$

then: $f(1) = 12 = g(1)$

$f(2) = 30 = g(2)$

$f(3) = 60 = g(3)$ hence $f = g$

i.e $f = \{(1,12), (2,30), (3,60)\} = g$

(What's the magic trick?

Observe: $f - g = x^3 - 6x^2 + 11x - 6$

$= (x-1)(x-2)(x-3)$

$= 0$ for $x \in \{1,2,3\}$)

Images

Def'n: Sps $f: A \to B$ is a function and $X \subseteq A$. The image of X under f, denoted $\mathrm{Im}_f(X)$, is defined by:

$$\mathrm{Im}_f(X) = \{b \in B \mid (\exists a \in X)(f(a) = b)\}$$

informally we could write this as: $\mathrm{Im}_f(X) = \{f(a) : a \in X\}$

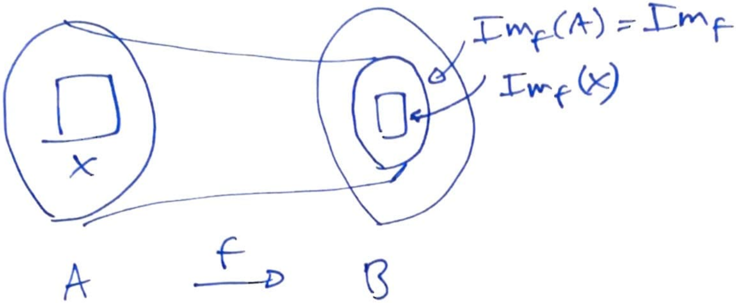
the point: $a \in X \Rightarrow f(a) \in \mathrm{Im}_f(X)$

Note: when $X = A$ itself we say $\mathrm{Im}_f(A)$ is the image of f and sometimes write $\mathrm{Im}f$.

so def'n says:- $Im_f(X)$ is the set of outputs of elements in X.

- $Im_f = Im_f(A)$ is the set of all outputs of f.

Pic



Ex's ① let $A = \{1,2,3\}$ $B = \{*, \heartsuit, \Delta\}$ and let $f: A \to B$ be the function $f = \{(1,*), (2,\heartsuit), (3,*)\}$

Then:- $Im_f(\{1,3\}) = \{f(1), f(3)\}$

$= \{*, *\} = \{*\}$

- and $Im_f = Im_f(\{1,2,3\}) = \{f(1), f(2), f(3)\}$

$= \{*, \heartsuit, *\}$

$= \{*, \heartsuit\}$

② Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$

Then: $Im_f(\{-1,0,1\}) = \{(-1)^2, 0, 1^2\}$

$= \{0, 1\}$

- $Im_f = \{x \in \mathbb{R} \mid x \geq 0\}$

Functions add a layer of complexity to the basic set theory of $\cup, \cap, \ldots$ we studied before.

Prop'n Sps $f: A \to B$ is a function and $S, T \subseteq A$.

Then $Im_f(S \cap T) \subseteq Im_f(S) \cap Im_f(T)$

Pf: - fix $y \in Im_f(S \cap T)$
- then $\exists x \in S \cap T$ s.t. $f(x) = y$
- hence $x \in S$ and $x \in T$
- hence $f(x) \in Im_f(S)$ and $f(x) \in Im_f(T)$
- i.e. $y \in Im_f(S)$ and $y \in Im_f(S)$

Since y was arbitrary the prop'n is proved.