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Ex: Q: for which $n \in \mathbb{N}$ do we have $2^n < n!$? Let's see...

n	2^n	$n!$
1	2	1
2	4	2
3	8	6
4	16	24
5	32	120
$\vdots$	$\vdots$	$\vdots$

Seems like: if $n \geq 4$ then $n! > 2^n$

Let's prove.

Prop'n For every $n \in \{4, 5, 6, \dots\} = \{n \in \mathbb{N} \mid n \geq 4\}$ we have $n! > 2^n$

(here: $n_0 = 4$, $S = \{4, 5, 6, \dots\}$)

Pf: Let $P(n)$ be the prop'n "$n! > 2^n$"

(BC) $P(4)$ holds since $4! > 2^4$

(IH) Fix $n \in \mathbb{N}$, $n \geq 4$ and assume $P(n)$ holds, i.e. assume $n! > 2^n$

note: we fix $n \geq 4$ not $n > 4$.

(IS) Then:

$$(n+1)! = n!\,(n+1)$$
$$> 2^n (n+1) \quad \text{by IH}$$
$$> 2^n \cdot 2 \quad \text{since } n \geq 4 \text{ so } n+1 \geq 5$$
$$= 2^{n+1}$$

We've shown $(n+1)! > 2^{n+1}$, i.e. $P(n+1)$

By induction we've proved for every $n \geq 4$ (i.e. for $n \in \{4,5,6,\dots\}$) we have $n! > 2^n$.

Induction w/ Jumps:

- sometimes we want to prove $P(n)$, not for all n,
  - but when n is even,
  - or... when n is odd,
  - or... when n is a multiple of 3, etc.

→ can still argue inductively.

Thm Let $P(n)$ be a var. prop'n.

Fix $n_0 \in \mathbb{Z}$ and $k \in \mathbb{N}$

← "starting pt" "jump"

Let $S = \{n_0, n_0+k, n_0+2k, \ldots\}$

If we have ① $P(n_0)$

and ② $(\forall n \in S)(P(n) \Rightarrow P(n+k))$

Then $(\forall n \in S)(P(n))$ holds.

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e.g. if $S = \{2, 4, 6, \ldots\}$

and we can show

① $P(2)$

② If $P(n)$, then $P(n+2)$

Then we've proved $P(n)$ holds $\forall n \in S$.

Ex Consider the alternating sum of the first n squares:

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1} n^2$$

$$= \sum_{k=1}^{n} (-1)^{k-1} k^2$$

Prop'n ① if n is odd we have:

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k \qquad \left(= \frac{n(n+1)}{2} \text{ by before}\right)$$

② if n is even we have:

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = -\sum_{k=1}^{n} k \qquad \left(= -\frac{n(n+1)}{2}\right)$$

Pf: ① here $n_0 = 1$ and jump $= 2$ so that $S = \{1, 3, 5, \dots\}$.

$P(n)$ is " $\sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k$ "

(BC) if $n = 1$, $\sum_{k=1}^{1} (-1)^{k-1} k^2 = 1^2 = 1 = \sum_{k=1}^{1} k$

So $P(1)$ holds.

(IH) Fix $n \in \{1, 3, 5, \dots\}$ and assume $P(n)$, i.e. $\sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k$

(IS) now consider the $n+2$ sum:

$$\sum_{k=1}^{n+2} (-1)^{k-1} k^2 = \sum_{k=1}^{n} (-1)^{k-1} k^2 + (-1)^{n} (n+1)^2 + (-1)^{n+1} (n+2)^2$$

$$= \sum_{k=1}^{n} (-1)^{k-1} k^2 - (n+1)^2 + (n+2)^2$$

since n odd

IH

$$= \sum_{k=1}^{n} k - (n+1)^2 + (n+2)^2$$

$$= \sum_{k=1}^{n} k + [\underbrace{(n+2) - (n+1)}_{1}][(n+2) + (n+1)]$$

$$= \sum_{k=1}^{n} k + (n+1) + (n+2) = \sum_{k=1}^{n+2} k$$

So $P(n+2)$ holds. By induction we've proved, $\forall n \in \{1,3,5,\dots\}$

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k$$

<u>Summary</u>: we showed

① $P(1)$ holds

② if $n \in \{1,3,5,\dots\}$ then $P(n) \Rightarrow P(n+2)$

It follows: $P(n)$ holds $\forall n \in \{1,3,5,\dots\}$.

② For n even. a similar arg. You try!

Outline is: ① verify $P(2)$, i.e.

$$\sum_{k=1}^{2} (-1)^{k-1} k^2 = -\sum_{k=1}^{2} k$$

② Fix $n \in \{2,4,6,\dots\}$, and assume

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = -\sum_{k=1}^{n} k$$

③ Use this to prove

$$\sum_{k=1}^{n+2} (-1)^{k-1} k^2 = -\sum_{k=1}^{n} k$$

<u>Fibonacci</u> <u>Sequence</u>: is defined recursively by:

$$f_0 = 0 \qquad f_1 = 1$$

$$f_n = f_{n-2} + f_{n-1} \qquad \text{for } n \geq 2.$$

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

$$f_0 \; f_1 \; f_2 \; f_3 \; f_4 \; \ldots$$

↪ Fib. sequence is a playground for inductive proofs.

<u>Prop'n</u> $\forall n \in \mathbb{N}$, we have:

$$\sum_{k=1}^{n} f_k = f_{n+2} - 1$$

(i.e. $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$)

<u>Pf</u>: (BC) if $n=1$ we have:

$$\sum_{k=1}^{1} f_k = f_1 = 1 = 2-1 = f_3 - 1 \checkmark$$

(IH) Fix $n \in \mathbb{N}$ and assume

$$\sum_{k=1}^{n} f_k = f_{n+2} - 1$$