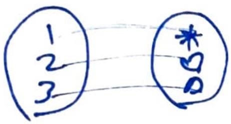
Infinity

The concept of cardinality:

- We would say that the set $\{*, \heartsuit, \triangle\}$ has 3 elements or is of size 3
- Why? By counting it!

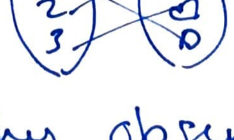
$$\begin{matrix} \{* , & \heartsuit, & \triangle\} \\ 1 & 2 & 3 \end{matrix}$$

- in doing so we are implicitly defining a bijection between $\{1,2,3\}$ and $\{*, \heartsuit, \triangle\}$



- we could've counted differently:

$$\begin{matrix} \{* , & \heartsuit, & \triangle\} \\ 2 & 3 & 1 \end{matrix}$$



- generalizing this observation:
we'll say two sets have the same size iff there is a bijection between them.

<u>Def'n</u> Two sets A, B have the same <u>cardinality</u> iff there is a bijection $f: A \to B$. In this case we write $A \sim B$.

<u>Note</u>: – in set theory courses, one defines, for every set A, the cardinal number $|A|$
– can then prove: $A \sim B$ iff $|A| = |B|$
(e.g. $|\{*, \bigcirc, \triangle\}| = |\{1,2,3\}| = 3$)
– defining cardinal #s beyond our scope:
for us $|A| = |B|$ just means $A \sim B$.
i.e. $\exists f: A \to B$ a bijection

<u>Properties of $\sim$</u>: ① For any set A, $id_A : A \to A$ is a bijection (Why?)

– Hence $A \sim A$, i.e. $\sim$ is reflexive

② If $f: A \to B$ is a bijection then f is invertible and $f^{-1}: B \to A$ is a bijection too (Why?)

– Hence if $A \sim B$ then $B \sim A$

– i.e $\sim$ is symmetric

3 On HW you proved:

if $f: A \to B$
$g: B \to C$ are bijections

then $g \circ f: A \to C$ is a bijection too.

- Hence if $A \sim B$ and $B \sim C$ then $A \sim C$
- i.e. $\sim$ is transitive

$\to$ 1 + 2 + 3 show $\sim$ is an equiv. relation on sets!

$\to$ $\sim$ most interesting when the sets being compared are <u>infinite</u>.

<u>Ex's</u> 1 $\{1,2,3\} \sim \{*, \heartsuit, \Delta\}$ since $f = \{(1,*), (2,\heartsuit), (3,\Delta)\}$ is a bijection.

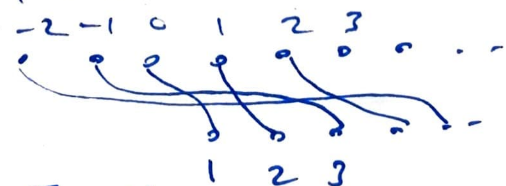
2 Let $-\mathbb{N}$ denote the set $\{-1, -2, \dots\}$

Define $f: \mathbb{N} \to -\mathbb{N}$ by $f(n) = -n$

- ez to check: f is a bijection
- Hence $\mathbb{N} \sim -\mathbb{N}$

3 We showed before: $f: \mathbb{Z} \to \mathbb{N}$ defined by:

$$f(n) = \begin{cases} 2n & n > 0 \\ 2(-n)+1 & n \leq 0 \end{cases}$$



is a bijection. Hence $\mathbb{Z} \sim \mathbb{N}$

④ Combining ② and ③ gives $\mathbb{Z} \sim \mathbb{N}$, by transitivity of $\sim$ (72)

<u>Def'n</u>: Let A, B be sets

① We write $A \lesssim B$ (or $|A| \leq |B|$) to mean: there is an <u>injection</u> $f: A \to B$

② We write $A \gtrsim B$ (or $|A| \geq |B|$) to mean: there is a surjection $g: A \to B$.

$\hookrightarrow$ We write $A < B$ to mean: there is an injection $f: A \to B$ but no bijection. (i.e. $A \lesssim B$ but $A \not\sim B$)

$\hookrightarrow$ Similarly $A > B$ means there is a surjection $f: A \to B$ but no bijection $g: A \to B$.

$\boxed{\text{Note!}}$ $A \gtrsim B$ is not simply the "reverse of" $A \lesssim B$, i.e. it is <u>not</u> asserting there is an injection from B to A. But this follows!

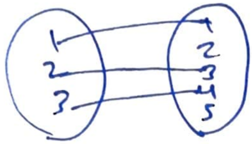
<u>Theorem</u> For all sets A, B we have:

$$A \lesssim B \text{ iff } B \gtrsim A$$

i.e. there is an injection $f: A \to B$ iff there is a surjection $g: B \to A$.

Before proving theorem, let's illustrate the ideas of proof w/ examples (173)

<u>Ex</u>: ① Consider $f: [3] \to [5]$ defined by $f(1) = 1$ $f(2) = 3$ $f(3) = 4$

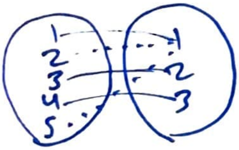


<u>Observe</u>: f is an injection, hence $[3] \lesssim [5]$

<u>idea</u>: to show $[5] \gtrsim [3]$, i.e. to get a surjection $g: [5] \to [3]$ we can "reverse" f, then map anything left over to something arbitrary.

e.g. let
$$\left.\begin{array}{l} g(1) = 1 \\ g(3) = 2 \\ g(4) = 3 \end{array}\right\} \text{"reverse" of } f$$

$g(2) = 1$
$g(5) = 1$
could put anything here



then: g is a <u>function</u> since f was injective (why?) and is clearly surjective.

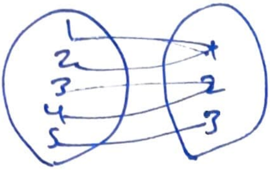
more generally can prove: if $A \lesssim B$ then $B \gtrsim A$ using this idea.

② Consider $g: [5] \to [3]$ defined by: 174

$g(1) = g(2) = 1$

$g(3) = g(4) = 2$

$g(5) = 3$

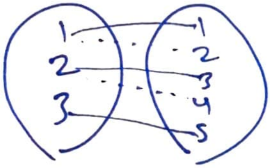


Observe: g is a surjection hence $[5] \gtrsim [3]$

Idea: to show $[3] \lesssim [5]$, i.e. to get an injection $f: [3] \to [5]$ we take some "reverse" of g w/o repeats

e.g. define $f: [3] \to [5]$ by:

$f(1) = 1 \quad f(2) = 3 \quad f(3) = 5$



f "reverses" g w/o repeats

then: f is a function because g was surjective and we deleted repeats (why?)

and it's injective because g was a function (why?)

more generally can prove: if $A \gtrsim B$ then $B \lesssim A$ using same idea.