Let's prove theorem:

<u>Pf</u> ($\Rightarrow$) Sps $A \lesssim B$, i.e. $\exists f: A \to B$ an injection.

<u>WTS</u>: $B \gtrsim A$, i.e. $\exists g: B \to A$ a surjection

Define g as follows:

- first, fix $a_0 \in A$
- now consider a given $b \in B$

↳ if $b \in \mathrm{Im} f$, then $\exists a \in A$ s.t. $f(a) = b$
And thus a is <u>unique</u> since f is injective
In this case define $g(b) = a$

↳ if $b \notin \mathrm{Im} f$ define $g(b) = a_0$

<u>Then</u>: ① g is a function from B to A

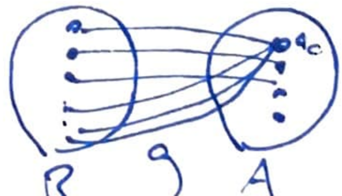
<u>Why</u>: if $b \in \mathrm{Im} f$ then $g(b) =$ unique $a \in A$ s.t. $f(a) = b$

if $b \notin \mathrm{Im} f$ then $g(b) = a_0$

② g is surjective:

<u>Why</u>: Given $a \in A$, let $b = f(a)$
Then $g(b) = a$, by defn!

<u>Picture</u>:



($\Leftarrow$) Now suppose $B \gtrsim A$, i.e.
$\exists g: B \to A$ a surjection.

<u>WTS</u>: $A \lesssim B$ i.e. there is $f: A \to B$ an injection

Define f as follows:

- given $a \in A$, since g is surjective there is at least one $b \in B$ s.t. $g(b) = a$, i.e. $\mathrm{PreIm}_g(\{a\}) \neq \emptyset$.
- so: pick <u>one</u> distinguished el't $b_a \in \mathrm{PreImg}_g(\{a\})$ and define $f(a) = b_a$.

(so: note: $g(b_a) = a$)

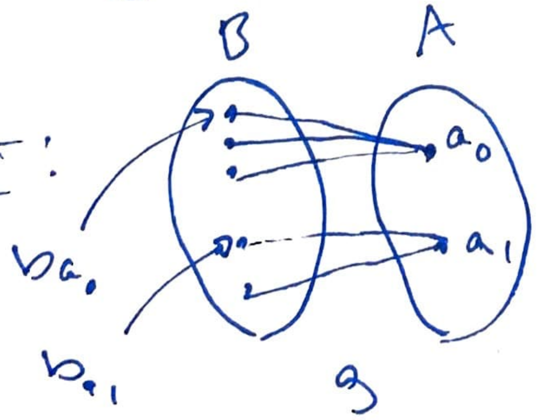
Now do this for every $a \in A$.

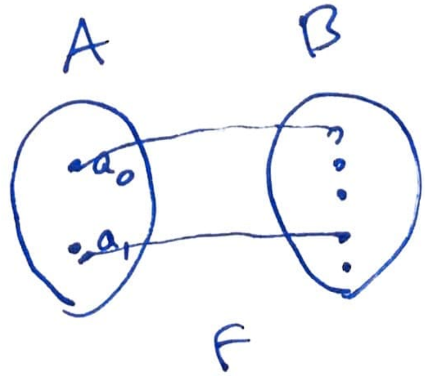
<u>Then</u>: ① f is a function from A to B

<u>Why</u>: every $a \in A$ has a unique output $b_a \in B$, from $\mathrm{PreImg}_g(\{a\})$

② f is injective: if $f(a) = f(a') = b$ then by def'n of f we have $g(b) = a$ and $g(b) = a'$. Hence $a = a'$, since g is a function

<u>Pict</u>:





✓

Properties of $\lesssim$ and $\gtrsim$

Sps A, B, C are sets

① $A \lesssim A$ and $A \gtrsim A$ since $id_A : A \to A$ is both an injection and surjection. Hence $\lesssim$ and $\gtrsim$ are reflexive.

② If $A \lesssim B$ and $B \lesssim C$ then $\exists f : A \to B$ and $g : B \to C$ injections. By HW: $g \circ f : A \to C$ is an injection. Hence $A \lesssim C$

Similarly if $A \gtrsim B \gtrsim C$ then $A \gtrsim C$.

So $\lesssim$ and $\gtrsim$ are transitive.

③ Are $\lesssim$ and $\gtrsim$ antisymmetric?

Not literally. ~~e.g. consider~~ If $A \lesssim B$ and $B \lesssim A$ not necessarily $A = B$. E.g. consider $\{*, \triangleleft, \triangle\}$ and $\{1, 2, 3\}$.

$\boxed{\text{But!}}$ We have the following:

<u>Theorem</u> (Cantor-Schroeder-Bernstein) (CSB)

If there exist <u>injections</u> $f_0 : A \to B$ and $f_1 : B \to A$ then there exists a <u>bijection</u> $f : A \to B$, i.e.

$$A \lesssim B \text{ and } B \lesssim A \implies A \sim B$$

Pf of theorem is a bit tricky – we'll skip and take for granted.

Combining CSB w/ previous theorem also gives: if $A \gtrsim B$ and $B \gtrsim A$ then $A \sim B$

So: $\lesssim$ and $\gtrsim$ are "antisymmetric up to $\sim$"

Some paradoxes of infinity:

Theme: $A \sim B$ when A, B are infinite can be counterintuitive!

① Let $E = \{2, 4, 6, 8, \ldots\}$

Then: $\mathbb{N} \sim E$ ("There are as many even numbers as whole numbers")

Why: $f : \mathbb{N} \to E$ $f(n) = 2n$ is a bijection

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & \cdots \\ f & \downarrow & \downarrow & \downarrow & \downarrow & \\ & 2 & 4 & 6 & 8 & \cdots \end{array}$$

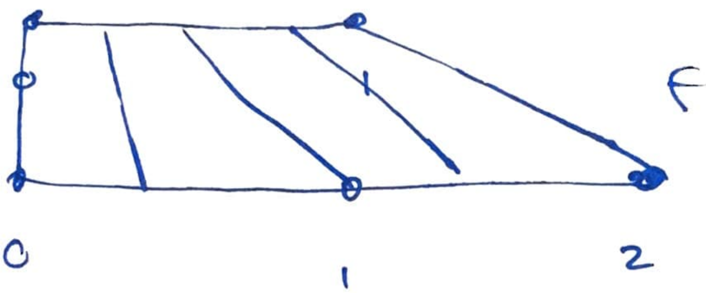
② Let $S = \{1, 4, 9, 16, \ldots\}$

Then: $\mathbb{N} \sim S$ ("there are as many squares as whole numbers")

Why: $f: \mathbb{N} \to S$ $f(n) = n^2$ is a bijection

③ Also: $[0,1] \sim [0,2]$ ("There are as many real numbers between 0 and 1 as between 0 and 2")

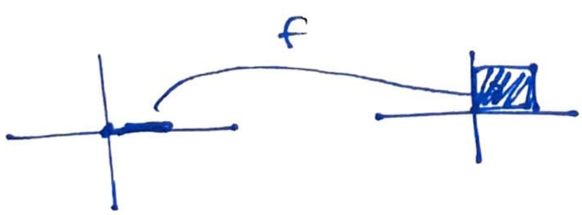
Why: $f: [0,1] \to [0,2]$
$f(x) = 2x$ is a bijection



④ And actually: "The side is as large as the square"

i.e. $[0,1] \sim [0,1] \times [0,1]$

i.e. $\exists f: [0,1] \to [0,1] \times [0,1]$ a bijection



Why: beyond our scope

Moral: infinite sets are wild!

Finiteness + Infiniteness

Def'n ① A set X is finite iff either

(i) $X = \emptyset$

or (ii) $\exists n \in \mathbb{N}$ and a bijection $f : [n] \to X$

② X is infinite iff it is not finite, i.e.

(i) $X \neq \emptyset$

and (ii) $\forall n \in \mathbb{N}$ there is no bijection $f : [n] \to X$

Following theorem says: $\mathbb{N}$ has "smallest" possible infinite size.

Theorem: if a set X is infinite then $\mathbb{N} \precsim X$

Pf: Sps X is infinite. We define an injection $f : \mathbb{N} \to X$ inductively

(BC) Since $X \neq \emptyset$ there is some $x_1 \in X$

let $f_1 = \{(1, x_1)\}$.

(IH) Sps at stage n we have defined a "partial injection" f_n, i.e.

$$f_n = \{(1,x_1), (2,x_2), \dots, (n, x_n)\}$$

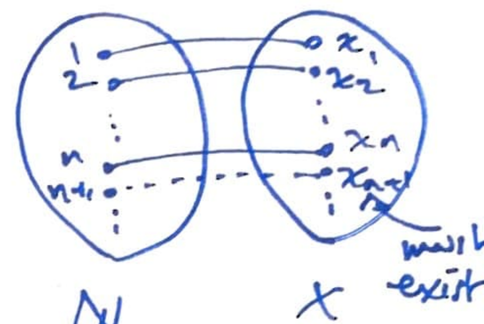
and the x_i's are all distinct.

(IS) We can view f_n as an injection from $[n]$ to X. Since X is not finite must be that f_n is <u>not</u> a surjection (o/w would be a bijection)

so: $\exists x_{n+1} \in X$ s.t. $x_{n+1} \notin \{x_1, .., x_n\}$

so define: $f_{n+1} = \{(1,x_1), (2,x_2), .., (n,x_n), (n+1,x_{n+1})\}$

$= f_n \cup \{(n+1, x_{n+1})\}$



Continuing: we may define injections $f_n : [n] \to X$ s.t. if $n \leq m$ then $f_n \subseteq f_m$

<u>Now</u>: let $f = \bigcup_{n \in \mathbb{N}} f_n = \{(1,x_1), (2,x_2), \dots\}$

then $f : \mathbb{N} \to X$ is an injection since we arranged the x_i's are distinct ✓

<u>Note</u>: proof is a <u>bit</u> informal; we've used not only induction but implicitly what set theorists call the <u>recursion theorem</u> and the axiom of choice.