__Note__ in general we don't have

$$\mathrm{Im}_f(S\cap T) = \mathrm{Im}_f(S) \cap \mathrm{Im}_f(T)$$

e.g. consider $f(x) = x^2$ on $\mathbb{R}$

Let $S = \{-1, 0\}$ $\quad T = \{0, 1, 2\}$

__Then__: $\mathrm{Im}_f(S) = \{f(-1), f(0)\} = \{(-1)^2, 0^2\}$
$= \{1, 0\}$

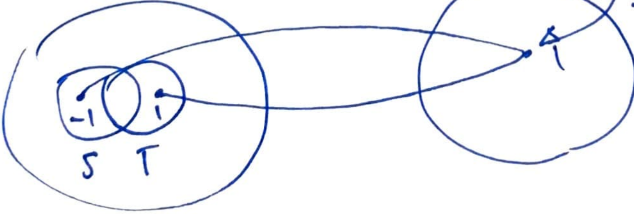
$\mathrm{Im}_f(T) = \{0, 1, 4\}$

$\Rightarrow \mathrm{Im}_f(S) \cap \mathrm{Im}_f(T) = \{0, 1\}$

But: $\mathrm{Im}_f(S\cap T) = \mathrm{Im}_f(\{0\})$
$= \{f(0)\} = \{0\}$

So in this case: $\mathrm{Im}_f(S\cap T) \subsetneq \mathrm{Im}_f(S) \cap \mathrm{Im}_f(T)$

The essence of the issue: functions can send multiple inputs to the same output. Schematically:



In $\mathrm{Im}_f(S) \cap \mathrm{Im}_f(T)$ but not $\mathrm{Im}_f(S\cap T)$!

Preimages

<u>Defn</u>: Sps $f: A \to B$ is a function and $Y \subseteq B$. The <u>preimage of Y under f</u>, denoted $\mathrm{PreIm}_f(Y)$, is defined as:

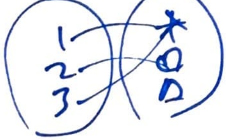
$$\mathrm{PreIm}_f(Y) = \{x \in A \mid f(x) \in Y\}$$

= the set of inputs whose outputs land in Y.

<u>Note</u>: since $f(x) \in B$ for every $x \in A$ we don't separately define $\mathrm{PreIm}_f(B)$ — this is always just A.

<u>Ex</u>: ① Let

$$A = \{1,2,3\}$$
$$B = \{*, \heartsuit, \Delta\}$$
$$f = \{(1,*), (2,\heartsuit), (3,*)\}$$



Then: $\mathrm{PreIm}_f(\{*\})$

$$= \{x \in A \mid f(x) \in \{*\}\}$$
$$= \{x \in A \mid f(x) = *\}$$
$$= \{1,3\}$$

$\mathrm{PreIm}_f(\{*, \triangledown\}) = \{x \in A \mid f(x) \in \{*, \triangledown\}\}$ (150)

$= \{1, 2, 3\} = A$

$\mathrm{PreIm}_f(\{\Delta\}) = \{x \in A \mid f(x) \in \{\Delta\}\}$

$= \phi.$

(2) Consider $f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

Then $\mathrm{PreIm}_f(\{0, 1\}) = \{x \in \mathbb{R} \mid f(x) \in \{0, 1\}\}$

$= \{x \in \mathbb{R} \mid x^2 \in \{0, 1\}\}$

$= \{-1, 0, 1\}$

$\mathrm{PreIm}_f([0, 2]) = \{x \in \mathbb{R} \mid x^2 \in [0, 2]\}$

$= \{x \in \mathbb{R} \mid 0 \leq x^2 \leq 2\}$

$= \{x \in \mathbb{R} \mid x^2 \leq 2\}$

$= \{x \in \mathbb{R} \mid -\sqrt{2} \leq x \leq \sqrt{2}\}$

$= [-\sqrt{2}, \sqrt{2}]$

$\mathrm{PreIm}_f([0, \infty)) = \{x \in \mathbb{R} \mid x^2 \in [0, \infty)\}$

$= \mathbb{R}$

Q: What happens if we take the preimage of the image of some $X \subseteq A$? Or the image of the preimage of some $Y \subseteq B$?

<u>Prop'n</u> Sps $f: A \to B$ is a function

(i) Fix $X \subseteq A$.

Then $\mathrm{PreIm}_f(\mathrm{Im}_f(X)) \supseteq X$

(ii) Fix $Y \subseteq B$

Then $\mathrm{Im}_f(\mathrm{PreIm}_f(Y)) \subseteq Y$.

<u>Pf</u>: (i) Fix $x \in X$.

By def'n: $\mathrm{PreIm}_f(\mathrm{Im}_f(X))$

$= \{y \in A \mid f(y) \in \mathrm{Im}_f(X)\}$

Since $x \in X$, we know $f(x) \in \mathrm{Im}_f(X)$ by def'n of $\mathrm{Im}_f(X)$

Hence $x \in \mathrm{PreIm}_f(\mathrm{Im}_f(X))$

Since x was arbitrary, (i) is proved.

(ii) Now fix $y \in \mathrm{Im}_f(\mathrm{PreIm}_f(Y))$

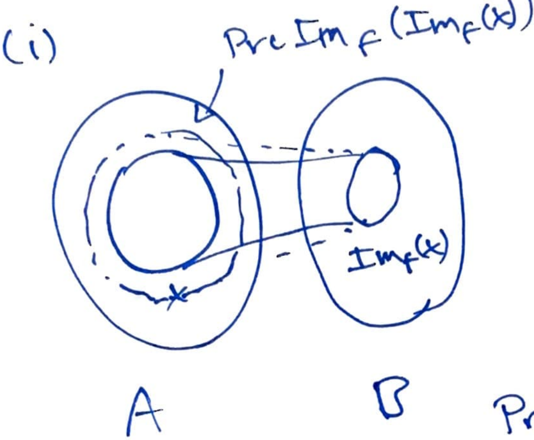
By def'n of image, $\exists x \in \mathrm{PreIm}_f(Y)$ s.t. $f(x) = y$

But then, by def'n of preimage, $f(x) \in Y$, i.e. $y \in Y$.

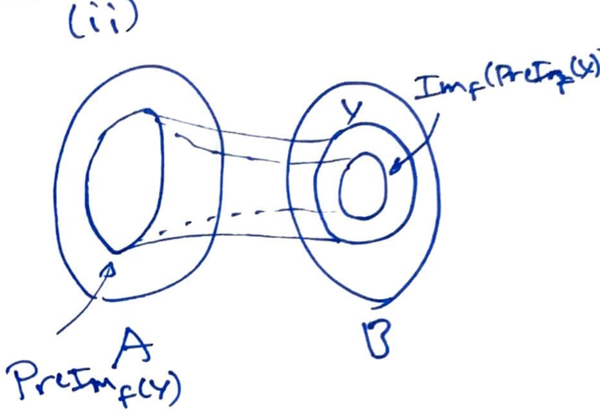
Since y was arbitrary, (ii) is proved ✓.

Pictures

(i)



(ii)



Note: in general neither containment can be reversed.

Ex: Define $f: \mathbb{R} \to \mathbb{R}$ by
$$f(x) = x^2$$

Let $X = \{1\}$

Then $\mathrm{Im}_f(X) = \mathrm{Im}_f(\{1\})$
$$= \{f(1)\}$$
$$= \{1\}$$

So: $\text{PreIm}_f(\text{Im}_f(X)) = \text{PreIm}_f(\{1\})$
$= \{x \in \mathbb{R} \mid f(x) \in \{1\}\}$
$= \{x \in \mathbb{R} \mid x^2 \in \{1\}\}$
$= \{-1, 1\}$

So we have strict containment in this case:

$X = \{1\} \subsetneq \{-1, 1\} = \text{PreIm}_f(\text{Im}_f(X))$

Now: let $Y = \{-2, 1\}$

Then: $\text{PreIm}_f(Y) = \{x \in \mathbb{R} \mid f(x) \in \{-2, 1\}\}$
$= \{x \in \mathbb{R} \mid x^2 \in \{-2, 1\}\}$
$= \{-1, 1\}$

So: $\text{Im}_f(\text{PreIm}_f(Y)) = \text{Im}_f(\{-1, 1\})$
$= \{f(-1), f(1)\}$
$= \{1, 1\} = \{1\}.$

Again we have strict containment:

$\text{Im}_f(\text{PreIm}_f(Y)) = \{1\} \subsetneq \{-2, 1\} = Y$

-jections

let $A = \{1, 2, 3\}$

$B = \{*, ♡\}$

$C = \{1, 2\}$

$D = \{*, ♡, \Delta\}$

define $g : A \to B$

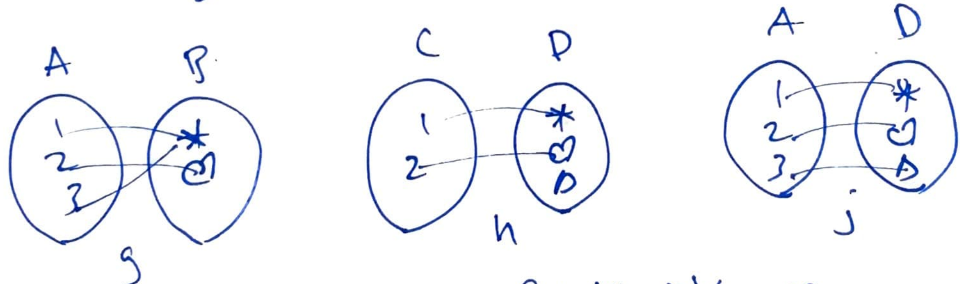
$h : C \to D$

$j : A \to D$

by: $g = \{(1, *), (2, ♡), (3, *)\}$

$h = \{(1, *), (2, ♡)\}$

$j = \{(1, *), (2, ♡), (3, \Delta)\}$



<u>Def'n</u> a function $f : X \to Y$ is <u>surjective</u> (or <u>onto</u>, or a <u>surjection</u>) iff $Im_f = Y$

i.e. iff $(\forall y \in Y)(\exists x \in X)(f(x) = y)$

e.g. g and j above are surjective, h is not since $\Delta \notin \operatorname{Im} h$

<u>Def'n</u> a function $f: X \to Y$ is <u>injective</u> (or <u>1-1</u>, or an <u>injection</u>) iff

$$(\forall x, y \in X)(f(x) = f(y) \Rightarrow x = y)$$

or, equivalently

$$(\forall x, y \in X)(x \neq y \Rightarrow f(x) \neq f(y))$$

"distinct inputs map to distinct outputs"

e.g. g above is <u>not</u> injective since $1 \neq 3$ but $g(1) = g(3) = *$

but h, j are injective.

<u>Def'n</u> a function $f: X \to Y$ is <u>bijective</u> (or a <u>bijection</u>) iff f is both injective and surjective.

e.g. - g above is <u>not</u> bijective (surjective, but not injective)

- nor is h (injective but not surjective)

- j <u>is</u> bijective