Thm says: $\mathbb{N}$ is "as small as possible" for an infinite set: even sets that appear smaller (e.g. E, O or other infinite subsets of $\mathbb{N}$) are actually not.

& OTOH: many sets which appear larger than $\mathbb{N}$ (e.g. $\mathbb{Z}$, $\mathbb{Q}$) are actually not too.

Def'n: A set X is countable iff $\mathbb{N} \sim X$.

Ex's ① $\mathbb{Z}$ is countable

Pf: we already showed

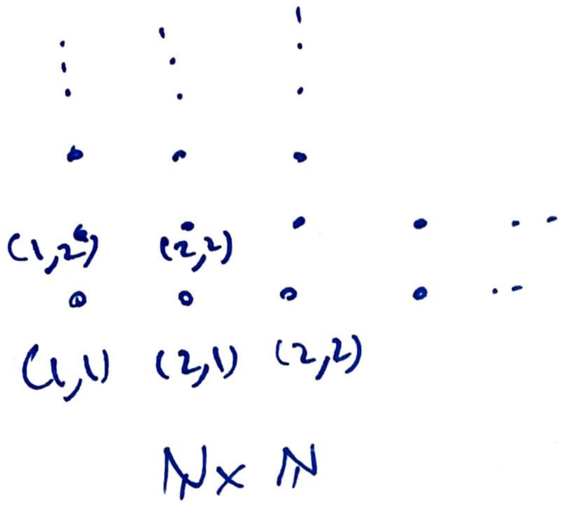
$$f: \mathbb{Z} \to \mathbb{N} \qquad f(n) = \begin{cases} 2n & n > 0 \\ 2(-n)+1 & n \leq 0 \end{cases}$$

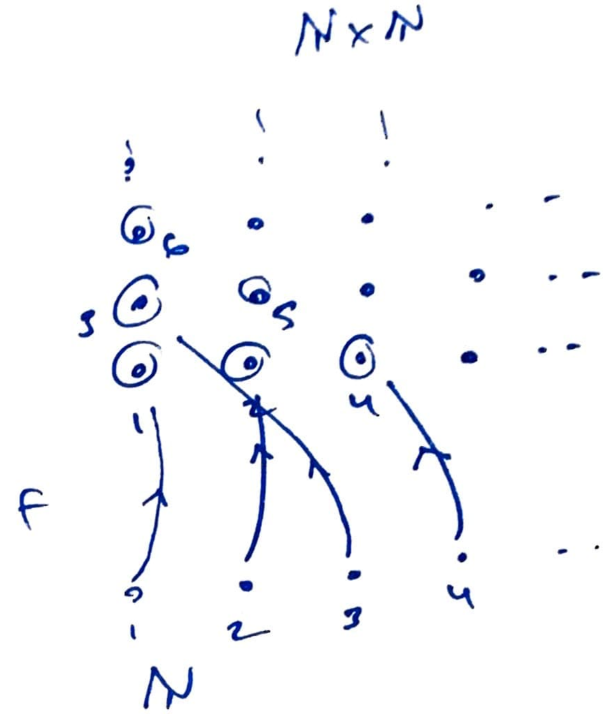
is a bijection.

② $\mathbb{N} \times \mathbb{N}$ is countable

Pf: need to construct a bijection $f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$

→ possible to do this explicitly (i.e. write down a fmla for f), but we just draw a picture to construct $f: \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ we "count $\mathbb{N}\times\mathbb{N}$ along its diagonals"





e.g. $f(1) = (1,1)$
$f(2) = (2,1)$
$f(3) = (1,2)$ .. etc.

f so defined is injective + surjective.

③ **Theorem**: If A, B are countable sets, then $A \times B$ is countable.

Pf: Sps A, B are countable, i.e. we have bijections $f: \mathbb{N} \to A$ and
$g: \mathbb{N} \to B$

We knew from above: $\mathbb{N} \sim \mathbb{N}\times\mathbb{N}$
so if we can show: $\mathbb{N}\times\mathbb{N} \sim A\times B$
we'll be done (by transitivity of $\sim$)

Consider $F: \mathbb{N}\times\mathbb{N} \to A\times B$
defined by $F(n,m) = (f(n), g(m))$

<u>Claim</u> F is a bijection

<u>Pf</u> (surj.) – Fix $(a,b) \in A\times B$
- since f,g both surj. $\exists n,m \in \mathbb{N}$ s.t. $f(n) = a$ $g(m) = b$
- Hence $F(n,m) = (f(n), g(m)) = (a,b)$ ✓

(inj.) – if $F(n,m) = F(n', m')$
- then $(f(n), g(m)) = (f(n'), g(m'))$
- so: $f(n) = f(n')$, $g(m) = g(m')$
- since f,g injective this gives $n = n'$ and $m = m'$
- i.e. $(n,m) = (n', m')$ ✓

hence $\mathbb{N}\times\mathbb{N} \sim A\times B$
hence $\mathbb{N} \sim A\times B$, as desired ✓

<u>Theorem</u>: $\mathbb{Q}$ is countable, i.e. $\mathbb{N} \sim \mathbb{Q}$

<u>Pf</u>: we'll prove:

$$\mathbb{N} \overset{①}{\lesssim} \mathbb{Q} \overset{②}{\lesssim} \mathbb{Z} \times \mathbb{N} \overset{③}{\lesssim} \mathbb{N}$$

By transitivity this gives $\mathbb{N} \lesssim \mathbb{Q} \lesssim \mathbb{N}$ which by CSB gives $\mathbb{N} \sim \mathbb{Q}$

① holds by our previous theorem, since $\mathbb{Q}$ is infinite. Alternately $f: \mathbb{N} \to \mathbb{Q}$ $f(n) = n$ is an injection.

③ holds by the preceding theorem we know $\mathbb{Z} \sim \mathbb{N}$ and $\mathbb{N} \sim \mathbb{N}$, hence $\mathbb{Z} \times \mathbb{N} \sim \mathbb{N}$

So remains to prove ②. We prove $\gtrsim$ version.

<u>Claim</u>: $\mathbb{Z} \times \mathbb{N} \gtrsim \mathbb{Q}$.

Pf: $F: \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ defined by $F(m,n) = \frac{m}{n}$ is surjective (why: given $q \in \mathbb{Q}$, if $q = \frac{m}{n}$ then $F(m,n) = q$!)

Note: F is <u>not</u> injective, e.g. $F(1,2) = F(2,4) = \dots$

Who cares! We've still shown $\mathbb{Z}\times\mathbb{N} \gtrsim \mathbb{Q}$.
Hence $\mathbb{Q} \lesssim \mathbb{Z}\times\mathbb{N}$.
As noted, this finishes proof that
$\mathbb{Q} \sim \mathbb{N}$. ✓

<u>Discussion</u>: - among infinite sets X, $\mathbb{N}$ is "small" in that $\mathbb{N} \lesssim X$ always.
- yet $\mathbb{N}$ is "large" in that for many sets X which appear larger than $\mathbb{N}$ (e.g. $\mathbb{Z}, \mathbb{Q}$) we actually have $X \sim \mathbb{N}$.
- Are there infinite sets X for which $\mathbb{N} < X$? Yes!

<u>Theorem</u> (Cantor): $\mathbb{N} < P(\mathbb{N})$

That is: $\mathbb{N} \lesssim P(\mathbb{N})$ but $\mathbb{N} \not\sim P(\mathbb{N})$

<u>Pf</u>: we knew $\mathbb{N} \lesssim P(\mathbb{N})$ since $P(\mathbb{N})$ is infinite (alternately: $f: \mathbb{N} \to P(\mathbb{N})$ $f(n) = \{n\}$ is an injection)

So we need to show

$\mathbb{N} \not\sim P(\mathbb{N})$