Proving surjectivity

ex: ① Define $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ by $f((m,n)) = m+n$

write $f(m,n)$

Claim: f is surjective

Pf: WTS: $(\forall x \in \mathbb{Z})(\exists (a,b) \in \mathbb{Z}\times\mathbb{Z})(f(a,b) = x)$

- so fix $x \in \mathbb{Z}$
- observe $f(0,x) = 0 + x = x$
- hence $\exists (a,b) \in \mathbb{Z}\times\mathbb{Z}$ s.t. $f(a,b) = x$ namely $(a,b) = (0,x)$
- since x was arbitrary, claim is proved. ✓

② Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x+1$

Claim: f is surjective.

Pf:
- fix $y \in \mathbb{R}$
- let $x = \frac{y-1}{2}$
- then $f(x) = f\left(\frac{y-1}{2}\right) = 2\left(\frac{y-1}{2}\right)+1 = y$
- since y was arbitrary, claim is proved.

③ Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$

Claim f is not surjective.

Pf: WTS $\neg (\forall y \in \mathbb{R})(\exists x \in \mathbb{R})(f(x) = y)$

i.e. $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(f(x) \neq y)$

Let $y = -1$. Then fix $x \in \mathbb{R}$.

Observe $f(x) = x^2 \geq 0 > -1$

Hence $f(x) \neq -1$

Since x was arbitrary we have

$(\forall x \in \mathbb{R})(f(x) \neq -1)$

i.e. $-1 \notin \operatorname{Im} f \Rightarrow f$ is not surjective

∴

Proving injectivity

Two approaches: fix $x, y \in A$ and either:

① Assume $f(x) = f(y)$, prove $x = y$

② Assume $x \neq y$, prove $f(x) \neq f(y)$

Ex's: ① Consider again $f: \mathbb{R} \to \mathbb{R}$

$f(x) = 2x + 1$.

<u>Claim</u> f is injective

<u>Pf</u>: - fix $x, y \in \mathbb{R}$

- assume $f(x) = f(y)$

i.e. $2x+1 = 2y+1$

$\Rightarrow$ $2x = 2y$

$\Rightarrow$ $x = y$

since x, y were arbitrary, claim is proved.

② Define $f: \mathbb{N} \to \mathbb{N}$ by $f(n) = n^2$

<u>Claim</u>: f is injective

<u>Pf</u>: Fix $n, m \in \mathbb{N}$ and sps $n \neq m$.

<u>WTS</u>: $f(n) \neq f(m)$, i.e. $n^2 \neq m^2$.

Two cases: ① $m < n$

② $n < m$

if ①: since n, m both <u>positive</u>, can square both sides of inequality to get: $m^2 < n^2$

so in particular $m^2 \neq n^2$, i.e. $f(m) \neq f(n)$

if ②: similar

since n, m were arbitrary, claim is proved.

③ Define $f: \mathbb{Z} \to \mathbb{Z}$ by $f(n) = n^2$

<u>Claim</u>: f is <u>not</u> injective since

e.g. $-2 \neq 2$ but $f(-2) = f(2) = 4$ ✓

<u>Proving bijectivity</u>

<u>Ex's</u> ① Consider again $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x+1$.

<u>Claim</u>: f is bijective

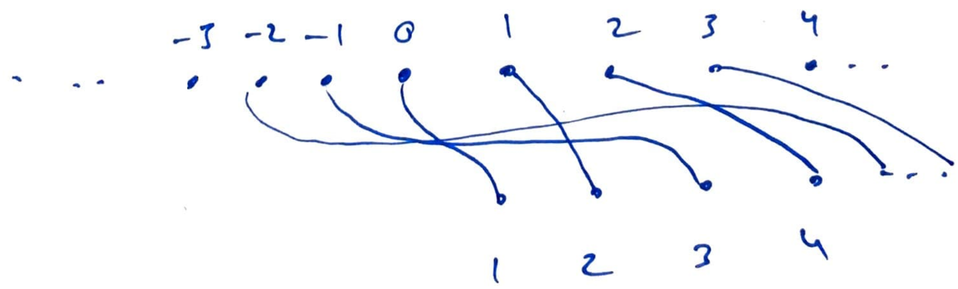
<u>Pf</u>: we've proved already f is both injective and surjective ✓

② <u>A spicy one</u>:

define $f: \mathbb{Z} \to \mathbb{N}$ by:

$$f(n) = \begin{cases} 2n & \text{if } n > 0 \\ 2(-n)+1 & \text{if } n \leq 0 \end{cases}$$

<u>Pic</u>:



Claim: f is bijective

Pf: (surjectivity)

- Fix $n \in \mathbb{N}$
- if n is even, then $n = 2k$ for some $k \in \mathbb{N}$ (so $k > 0$)
- hence $f(k) = 2k = n$ (by def'n of f)
- if n is odd then $n = 2k+1$ for some $k \in \mathbb{N} \cup \{0\}$
- hence $k \geq 0$, so $-k \leq 0$
- hence $f(-k) = 2k+1 = n$ (by def'n of f)
- in either case $(\exists x \in \mathbb{Z})(f(x) = n)$
- hence f is surjective ✓

(injectivity)

- Fix $n, m \in \mathbb{Z}$ and assume $n \neq m$

we WTS $f(n) \neq f(m)$

- We may assume $n < m$ since case when $m < n$ is similar.

Case 1: $0 < n < m$

- then $f(n) = 2n < 2m = f(m)$
- so $f(n) \neq f(m)$

Case 2 : $n < m \leq 0$

- then $f(n) = 2(-n) + 1$
$f(m) = 2(-m) + 1$

- since $n < m$, we have $-n > -m$
$\Rightarrow 2(-n)+1 > 2(-m)+1$
$\Rightarrow f(n) > f(m)$
$\Rightarrow f(n) \neq f(m)$

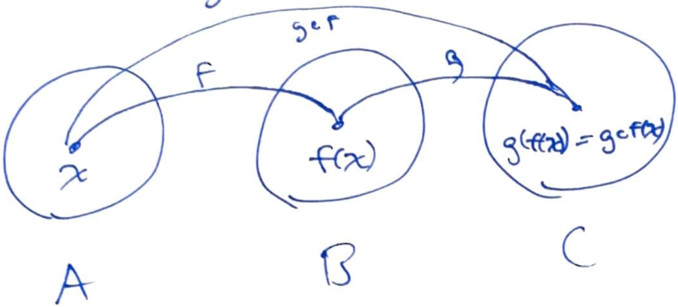
Case 3 $n \leq 0 < m$
- then $f(n) = 2(-n) + 1$ is odd
$f(m) = 2m$ is even
- hence $f(n) \neq f(m)$ in this case as well

- hence in all cases $f(n) \neq f(m)$
- since n, m were arbitrary, injectivity is proved
→ hence f is bijective.

Compositions

Def'n Sps $f: A \to B$ and $g: B \to C$ are functions. The composition of f and g, denoted $g \circ f$, is defined by:

$$(\forall x \in A) \quad g \circ f(x) = g(f(x))$$



So defined, $g \circ f$ is a function from A to C.

Ex's: Define $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by

$f(m,n) = m+n$

$g: \mathbb{Z} \to \mathbb{N}$ by

$g(n) = n^2 + 1$.

then $g \circ f(1,3) = g(f(1,3))$

$= g(1+3) = g(4)$

$= 4^2 + 1 = 17$