in general: $g \circ f(m,n) = g(f(m,n))$
$$= g(m+n)$$
$$= (m+n)^2 + 1$$

observe: $g \circ f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$

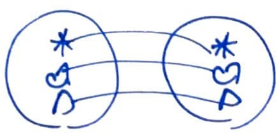
The identity function

Def'n Sps A is a set. The identity function on A, denoted id_A, is defined by:

$$id_A : A \to A$$
$$id_A(x) = x$$

e.g. if $A = \{*, ♡, \Delta\}$ then $id_A : A \to A$ is:

$$id_A = \{(*,*), (♡,♡), (\Delta,\Delta)\}$$



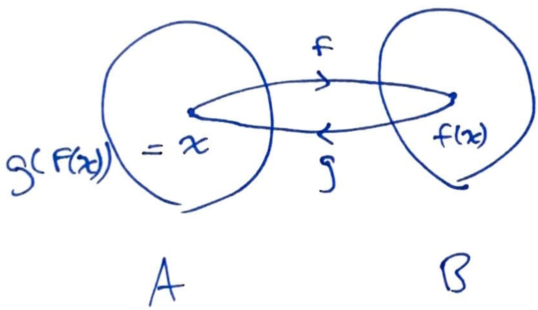
Def'n a function $f: A \to B$ is called invertible iff there is a function $g: B \to A$ s.t.:

$$g \circ f = id_A \quad \text{i.e. } (\forall x \in A)\ g(f(x)) = x$$

and

$$f \circ g = id_A \quad \text{i.e. } (\forall y \in B)\ f(g(y)) = y$$

If such g exists it is called the inverse of f and denoted f^{-1}



Ex. Consider $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$

Let $g: \mathbb{R} \to \mathbb{R}$ be defined by:

$$g(x) = \frac{x-1}{2}$$

Observe: $(\forall x \in \mathbb{R})$ $g(f(x)) = g(2x+1)$

$$= \frac{2x+1-1}{2}$$

$$= x$$

hence $g \circ f = id_{\mathbb{R}}$

also: $(\forall x \in \mathbb{R})$ $f(g(x)) = f\left(\frac{x-1}{2}\right)$

$$= 2\left(\frac{x-1}{2}\right) + 1$$

$$= x$$

hence $f \circ g = id_{\mathbb{R}}$.
thus: f is invertible and it's inverse f^{-1} is g.

Note: not all functions are invertible

In fact:

Theorem: a function $f: A \to B$ is invertible iff f is a bijection

Pf: $(\Rightarrow)$ Sps f is invertible and let $g = f^{-1}$ be its inverse.

WTS: f is a bijection

(surjectivity) - fix $y \in B$

- let $x = g(y)$
- then $f(x) = f(g(y)) = y$

(since $f \circ g = id_B$)

Since $y \in B$ was arbitrary, f is surjective

(injectivity) - fix $x, y \in A$ and sps $f(x) = f(y)$

- then $g(f(x)) = g(f(y))$
- hence $x = y$ (since $g \circ f = id_A$)

Since $x, y \in A$ were arbitrary, f is injective.

($\Leftarrow$) Sps now that $f : A \to B$ is a bijection

WTS: f is invertible.

Define: $g = \{(b,a) \in B \times A \mid (a,b) \in f\}$

Claim: ① g is a function from ~~A~~ B to A

② $g = f^{-1}$

Pf of ①: WTS $(\forall b \in B)$ $\exists$ a unique $a \in A$ s.t. $(b,a) \in g$.

- so fix $b \in B$

(existence) - since f is surjective, $\exists a \in A$ s.t. $f(a) = b$, i.e. $(a,b) \in f$

- hence $(b,a) \in g$, by def'n of g

(uniqueness) - spse there is $a' \in A$ s.t. $(b,a') \in g$ as well.

- then $(a',b) \in f$
- i.e. $f(a') = b$
- but then $f(a') = f(a)$. Since f is injective, must have $a' = a$ ✓

Hence ① is proved ✓

Pf of ②: WTS: $g = f^{-1}$ i.e. $g \circ f = id_A$
$f \circ g = id_B$

- so fix $a \in A$
- let $b = f(a)$ so that $(a,b) \in f$
- then $(b,a) \in g$ i.e. $g(b) = a$
- i.e. $g(f(a)) = a$
- since $a \in A$ was arbitrary, $g \circ f = id_A$

- Now fix $b \in B$. Let $a = g(b)$
  so that $(b,a) \in g$
- then must be $(a,b) \in f$, i.e. $f(a) = b$
- hence $f \circ g(b) = f(g(b)) = f(a) = b$
- since $b \in B$ was arbitrary, $f \circ g = id_B$.

Hence $g = f^{-1}$ as claimed.
Hence f is invertible ✓

We can sometimes use theorem to prove a given f is a bijection.

ex: Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x+1$

Claim: f is a bijection

Pf: we checked above that if $g(x) = \frac{x-1}{2}$
then $g = f^{-1}$
Hence f is invertible
By theorem, f is a bijection