② How many strings of 4 <u>or fewer</u> letters can be formed?

<u>Sol'n</u>: Let A be the set of such strings.

We can partition A as:

$$A = A_1 \cup A_2 \cup A_3 \cup A_4 \quad \text{pairwise disjoint}$$

where A_i = set of strings of length i.

By the rule of sum:

$$|A| = |A_1| + |A_2| + |A_3| + |A_4|$$

By rule of product:

$$|A_1| = 26 \quad |A_2| = 26 \cdot 26 \quad |A_3| = 26 \cdot 26 \cdot 26$$

$$|A_4| = 26 \cdot 26 \cdot 26 \cdot 26$$

So: $|A| = 26 + 26^2 + 26^3 + 26^4$

$= 475,254.$

<u>Permutations + Arrangements</u>

<u>Def'n</u>: Given a finite set A, a <u>permutation</u> of A is an ordered list of elements of A in which every element appears exactly once.

e.g. if $A = \{1,2,3\}$ then 213 and 321 are permutations of A; 23 and 2231 are not.

<u>Prop'n</u>: Fix $n \in \mathbb{N} \cup \{0\}$. If A has size n, then the # of permutations of A is $n!$

<u>Pf</u>: - if $|A| = 0$ then $A = \emptyset$ and there is $0! = 1$ permutation of A (the empty permutation)

- if $|A| = n \geq 1$ then a permutation

$$a_1 a_2 \cdots a_n$$

is formed uniquely by a sequence of n choices:

choose a_1, then a_2, ..., then a_n

$\downarrow$ n choices, $\downarrow$ $n-1$ choices .. $\downarrow$ 1 choice

$\Rightarrow$ By ROP: # of perms is $n(n-1) \cdots 1 = n!$

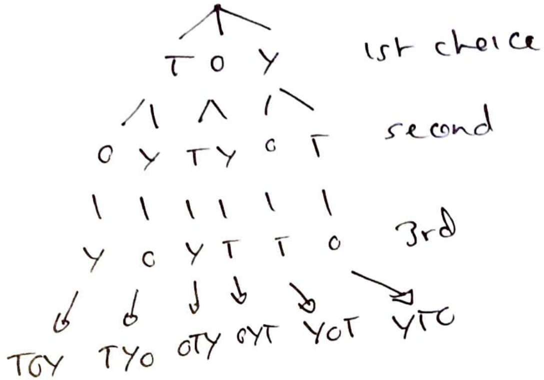
<u>Ex</u>: How many anagrams of the word TOY are there?

<u>Sol'n</u>: an anagram is just a permutation of $\{T, O, Y\}$. (Important! TOY has no repeated letters) By prop'n # of anagrams is $3!$

$= 6$

We can list them:

TOY OTY YTO
TYO OYT YOT



<u>Def'n</u>: Fix $k, n \in \mathbb{N} \cup \{0\}$ with $k \le n$. Given a set A w/ $|A| = n$, a <u>k-arrangement</u> of A is an ordered list of k elements of A w/ no repeats.

e.g. if $A = \{1,2,3,4,5,6\}$

then 254 and 152 are 3-arrangements of A; 115 and 29 are not k-arrangements of A (for any k)

<u>Prop'n</u> the # of k-arrangements of a set of size n is $\dfrac{n!}{(n-k)!} = n(n-1)\cdots(n-(k-1))$

Pf: a k-arrangement

$a_1 a_2 \cdots a_k$

is formed by making a sequence of k-choices

Choose a_1, then a_2, $\ldots$, then a_k

$\uparrow$ n choices $\qquad \uparrow$ $n-1$ $\qquad \cdots \qquad \uparrow$ $n-(k-1)$

$\hookrightarrow$ by ROP, # of k-arrangements

$$= n(n-1)\cdots(n-(k-1)) = \frac{n!}{(n-k)!}$$

Ex: How many 3-letter strings w/o repeats can be formed using the English alphabet? (e.g. AFG, WTS)

Sol'n: such strings are just 3-arrangements of $\{a, b, \ldots, z\}$

by prop'n, # of such strings

$$= 26 \cdot 25 \cdot 24 = 15,600$$

Selections

Def'n: Fix $n, k \in \mathbb{N} \cup \{0\}$ with $k \leq n$. Given A with $|A| = n$, a k-selection (or k-combination) is a subset of A of size k (i.e. an unordered list of k elements of A)

e.g. $\{2,3\}$ is a 2-selection of $\{1,2,3,4\}$

Notation $\binom{n}{k}$ denotes # of k-selections from a set of size n.

Prop'n Fix n, k with $k \leq n$.

Then $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

Pf: - Spse A is a set of size n and let S be the set of k-arrangements of A.

- We count $|S|$ in two ways
- We already know:

$$|S| = \frac{n!}{(n-k)!}$$

CTOIt: an element of S (i.e. a k-arrangement of A) can be formed in 2 steps: (218)

1st: choose a k-selection $\{a_1, \dots, a_k\}$ from A

2nd: choose an ordering (i.e. permutation) of this selection

So there are (by def'n) $\binom{n}{k}$ many ways to make 1st choice, and (by ROP) $k!$ ways to make 2nd.

Hence (by ROP): $|S| = \binom{n}{k} k!$

So: $\binom{n}{k} k! = \dfrac{n!}{(n-k)!}$

$$\Rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \checkmark$$

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e.g. $\binom{10}{3} = \dfrac{10!}{3!7!} = \dfrac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{3! \cancel{7!}}$

$= 10 \cdot 9 \cdot 8 / 3 \cdot 2 \cdot 1$

$= 120$

notice: $\binom{10}{7} = \dfrac{10!}{7!3!}$ too

In general: $\binom{n}{k} = \binom{n}{n-k}$