Ex: How many ways are there to choose a chief and two hench persons from a group of 10 ppl?

Sol'n: - 10 choices for chief

- once chief chosen, $\binom{9}{2}$ choices for hench ppl.

$\Rightarrow$ # of such committees $= 10\binom{9}{2}$

$$= 10 \cdot \frac{9!}{7!2!}$$

$$= 10 \cdot \frac{9 \cdot 8}{2 \cdot 1} = 360$$

Alternate sol'n: first pick 3 people for committee, then choose chief from among these:

$\Rightarrow$ # of such committees $= \binom{10}{3}\binom{3}{1}$

$$= 120 \cdot 3 = 360$$

as before!

Countin' Poker hands

- a **standard deck** consists of 52 cards
- each card has 1 of 4 possible **suits**:

$(\heartsuit, \diamondsuit, \clubsuit, \spadesuit)$

and 1 of 13 possible **ranks**:

$(A, 2, 3, \ldots, 9, 10, J, Q, K, A)$

- e.g. $A\diamondsuit$ and $9\heartsuit$ are cards.
- a **poker hand** is a 5-selection from a standard deck.

Ex's ① How many distinct hands are possible?

Sol'n: $\binom{52}{5} = 2{,}598{,}960.$

② A **full house** is a hand consisting of 3 cards of one rank and 2 cards of another, e.g. $A\diamondsuit, A\heartsuit, 3\spadesuit, 3\diamondsuit, 3\clubsuit$

How many distinct full house hands are possible?

Sol'n: - pick two ranks, $\binom{13}{2}$

- from these, pick the 3-card rank, $\binom{2}{1}$
- pick three cards from this rank, $\binom{4}{3}$
- pick two cards from the 2-card rank $\binom{4}{2}$

$\Rightarrow$ # of full house hands is:

$$\binom{13}{2}\binom{2}{1}\binom{4}{3}\binom{4}{2} = 3,744$$

3) A 3-of-a-kind consists of 3 cards from a single rank and 2 cards from two other distinct ranks

e.g. 3 Q's, a 10, and a J.

Q: How many 3-of-a-kind hands are possible?

Sol'n: - pick 3-card rank $\binom{13}{1}$

- from this rank, pick 3 cards $\binom{4}{3}$

- pick remaining two ranks $\binom{12}{2}$
- from the first, pick a card $\binom{4}{1}$
- also from the second $\binom{4}{1}$

$\Rightarrow$ # of 3 of a kind hands =

$$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1} = 54{,}912$$

<u>Alt sol'n</u>: - Pick three ranks $\binom{13}{3}$

- from these, pick 3-card rank $\binom{3}{1}$
- pick three cards from this rank $\binom{4}{3}$
- from other two ranks, pick cards

$\binom{4}{1}\binom{4}{1}$

But: $\binom{13}{3}\binom{3}{1}\binom{4}{3}\binom{4}{1}\binom{4}{1} = 54{,}912$

too ✓

<u>Binary sequences</u>: a <u>binary sequence</u> (of length n) is an ordered sequence of 0's and 1's (of length n)

e.g. 011 and 101 are binary sequences of length 3.

Let P_n denote the set of binary sequences of length n.

Ex: ① What is $|P_n|$?

② How many sequences $s \in P_n$ have at least two 1's? (assuming $n \geq 2$)

Sol'n: ① each $s \in P_n$ formed by making a sequence of n choices:

$$\underbrace{\underline{0/1}\ \underline{0/1}\ \cdots\ \underline{0/1}}_{\text{length } n}$$

2 choices for each space

$\Rightarrow |P_n| = 2 \cdot 2 \cdots 2$
$= 2^n$

② easier to count # of seq's w/ either zero 1's or one 1, then subtract:

\# w/ zero 1's $= 1$ (just $00\ldots0$)

\# w/ one 1 $= n$ (one for each place to put the 1)

$\Rightarrow$ # w/ at least two 1's

$= 2^n - n - 1$

<u>Theorem</u>: Fix $n \in \mathbb{N} \cup \{0\}$

Then: $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$

$= \sum_{k=0}^{n} \binom{n}{k}$

<u>Pf</u>: if $n=0$, then $2^0 = 1 = \binom{n}{0} = \sum_{k=0}^{0} \binom{n}{k}$ ✓

So sps $n \geq 1$.

We proved $|P_n| = 2^n$

We can partition $P_n = S_0 \cup S_1 \cup S_2 \cup \cdots \cup S_n$

where S_k = set of sequences w/ exactly k-many 1's.

<u>Observe</u>: $|S_k| = \binom{n}{k}$ = # of ways to pick k positions out of n where 1's appear.

Hence (by RoS):

$2^n = |P_n| = |S_0| + |S_1| + \cdots + |S_n|$

$= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$

$= \sum_{k=0}^{n} \binom{n}{k}$ ✓