

Ex: How many ways are there to choose a chief and two bench persons from a group of 10 ppl?

Sol'n: - 10 choices for chief
 - once chief chosen, $\binom{9}{2}$ choices for bench ppl.

$$\begin{aligned}\Rightarrow \# \text{ of such committees} &= 10 \binom{9}{2} \\ &= 10 \cdot \frac{9!}{7!2!} \\ &= 10 \cdot \frac{9 \cdot 8}{2 \cdot 1} = 360\end{aligned}$$

Alternate sol'n: first pick 3 people for committee, then choose chief from among these:

$$\begin{aligned}\Rightarrow \# \text{ of such committees} &= \binom{10}{3} \binom{3}{1} \\ &= 120 \cdot 3 = 360 \\ &\quad \text{as before!}\end{aligned}$$

Countin' Poker hands

(220)

- a standard deck consists of 52 cards
- each card has 1 of 4 possible suits:

(, , , )

and 1 of 13 possible ranks:

(A, 2, 3, ..., 9, 10, J, Q, K, A)

- e.g. $A\heartsuit$ and $9\heartsuit$ are cards.
- a poker hand is a 5-selection from a standard deck.

Ex's ① How many distinct hands are possible?

Sol'n: $\binom{52}{5} = 2,598,960.$

② A full house is a hand consisting of 3 cards of one rank and 2 cards of another, e.g. $A\heartsuit, A\heartsuit, 3\heartsuit, 3\heartsuit, 3\heartsuit$

How many distinct full house hands are possible?

Sol'n: - pick two ranks, $\binom{13}{2}$ (221)
 - from there, pick the 3-card rank, $\binom{2}{1}$
 - pick three cards from this rank, $\binom{4}{3}$
 - pick two cards from the 2-card rank $\binom{4}{2}$

$\Rightarrow$ # of full house hands is:

$$\binom{13}{2} \binom{2}{1} \binom{4}{3} \binom{4}{2} = 3,744$$

③ A 3-of-a-kind consists of 3 cards from a single rank and 2 cards from two other distinct ranks
 e.g. 3 Q's, a 10, and a J.

Q: How many 3-of-a-kind hands are possible?

Sol'n: - pick 3-card rank $\binom{13}{1}$
 - from this rank, pick 3 cards $\binom{4}{3}$

- pick remaining two ranks $\binom{12}{2}$ (222)
- from the first, pick a card $\binom{4}{1}$
- also from the second $\binom{4}{1}$

$\Rightarrow$ # of 3 of a kind hands =

$$\binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{1} \binom{4}{1} = 54,912$$

Alt sol'n : - Pick three ranks $\binom{13}{3}$

- from there, pick 3-card rank $\binom{3}{1}$
- pick three cards from this rank $\binom{4}{3}$
- from other two ranks, pick cards

$$\binom{4}{1} \binom{4}{1}$$

But: $\binom{13}{3} \binom{3}{1} \binom{4}{3} \binom{4}{1} \binom{4}{1} = 54,912$
too ✓

Binary sequences : a binary sequence

(of length n) is an ordered sequence
of 0's and 1's (of length n)

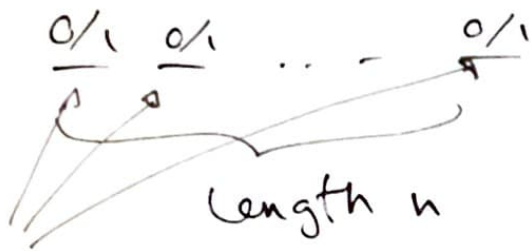
e.g. 011 and 101 are binary sequences of length 3. (223)

Let P_n denote the set of binary sequences of length n .

Ex: ① What is $|P_n|$?

② How many sequences $s \in P_n$ have at least two 1's? (assuming $n \geq 2$)

Sol'n: ① each $s \in P_n$ formed by making a sequence of n choices:



2 choices for each space

$$\Rightarrow |P_n| = 2 \cdot 2 \cdots 2 = 2^n$$

② easier to count # of seq's w/ either zero 1's or one 1, then subtract:

$$\# \text{ w/ zero 1's} = 1 \quad (\text{just } 00 \cdots 0)$$

$$\# \text{ w/ one 1} = n \quad (\text{one for each place to put the 1})$$

$$\Rightarrow \# w/ \text{ at least two } 1's \\ = 2^n - n - 1$$

(229)

Theorem: Fix $n \in \mathbb{N} \setminus \{0\}$

$$\text{Then: } 2^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} \\ = \sum_{k=0}^n \binom{n}{k}$$

$$\text{Pf: if } n=0, \text{ then } 2^0 = 1 = \binom{n}{0} = \sum_{k=0}^0 \binom{n}{k}$$

So sps $n \geq 1$.

We proved $|P_n| = 2^n$

We can partition $P_n = S_0 \cup S_1 \cup S_2 \cup \dots \cup S_n$

where S_k = set of sequences w/ exactly k -many 1's.

observe: $|S_k| = \binom{n}{k} = \# \text{ of ways to pick } k \text{ positions out of } n \text{ where } 1's \text{ appear.}$

Hence (by RoS):

$$2^n = |P_n| = |S_0| + |S_1| + \dots + |S_n| \\ = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} \\ = \sum_{k=0}^n \binom{n}{k} \quad \checkmark$$