Selection w/ repetition

Q: How many ways are there to select n objects from k types of objects, if repetition allowed.

Ex's: Dee's Donuts sells 4 types of donuts; you want to buy a dozen. How many distinct ways to do this?

Sol'n Imagine putting down 3 "spacers" to separate donut types.

ooo | oo | ooooo | oo

type 1 2 3 4

this "donut + spacer" diagram would correspond to purchase of:

3 donuts of type 1
2 " " " 2
5 " " " 3
2 " " " 4

→ can view diagram as a 01-sequence w/ 12 0's (for donuts) 3 1's (to separate 4 types)

→ conversely: any such sequence (12 0's, 3 1's) corresponds to a selection of 12 donuts:

→ e.g.

1 0 1 0 0 0 0 0 0 0 1 0 0 0 0

corresponds to:

0	tp1	donuts
1	tp2	"
7	tp3	"
4	tp4	"

→ Hence: # of ways to make a selection

= # of 01-seqs w/ 12 0's and 3 1's

= # of 01-seqs of length 15 w/ 3 1's

$= \binom{15}{3} = 455$

Same reasoning in general proves:

Theorem The # of ways to select n objects from k types w/ repetition allowed is:

$$\binom{n+(k-1)}{k-1}$$

(k-1 because we need k-1 "spacers" to separate k types of objects)

Ex: Sps we roll n (indistinguishable) 6-sided dice.

How many distinct outcomes are possible?

Sol'n: each of the n dice can roll into 6 possible "types"

1	2	3	4	5	6
oo	o		ooo	o	oo

hence # of possible outcomes is:

$$\binom{n+(6-1)}{6-1} = \binom{n+5}{5}$$

So if we roll 10 dice, # is:

$$\binom{15}{5} = 3003$$

Countin' in Two Ways:

Thm (Pascal's identity): Fix $n, k \in \mathbb{N}$ with $k \leq n$. Then:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Pf: Let S be the set of k element subsets of $[n] = \{1, 2, \ldots, n\}$

Then $|S| = \binom{n}{k}$

OTOH: We can partition S into S_1 and T

where: S_1 = k element subsets of $[n]$ that contain 1.

T = k el't subsets of $[n]$ that don't contain 1.

Then: $|S| = |S_1| + |T|$

Observe: subsets in S_1 are formed by selecting $k-1$ elements from $\{2, 3, \ldots, n\}$ (1 is auto included) $\Rightarrow |S_1| = \binom{n-1}{k-1}$

Subsets in T are formed by selecting k elements from $\{2, 3, \ldots, n\}$

$\Rightarrow |T| = \binom{n-1}{k}$

Hence: $|S| = \binom{n-1}{k-1} + \binom{n-1}{k}$

so $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Claim: Fix $n, k \in \mathbb{N}$, $k \leq n$. Then:

$$\binom{n}{k} k = n \binom{n-1}{k-1}$$

Pf: Let S denote set of committees of k ppl chosen from a group of n ppl, w/ a specified chairperson.

such a committee can be formed by:

- picking the committee members $\binom{n}{k}$
- from there, picking the chair $\binom{k}{1} = k$

$\Rightarrow |S| = \binom{n}{k} k$

or by:

- picking a chair first: $\binom{n}{1} = n$
- from remaining $n-1$ ppl, choose remaining $k-1$ committee members $\binom{n-1}{k-1}$

Hence: $|S| = n \binom{n-1}{k-1}$ too

$$\Rightarrow n \binom{n-1}{k-1} = \binom{n}{k} k$$

In this case can verify identity algebraically:

$$\binom{n}{k} k = \frac{n!}{k!(n-k)!} k$$

$$= \frac{n!}{(k-1)!\,(n-k)!}$$

and $$n\binom{n-1}{k-1} = n \frac{(n-1)!}{(k-1)!\,[(n-1)-(k-1)]!}$$

$$= \frac{n!}{(k-1)!\,(n-k)!}$$

same.

<u>Prop'n</u> Fix $n \in \mathbb{N}$. Then:

$$n 2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k$$

<u>Pf</u>: Let S be the set of nonempty committees w/ a chairperson chosen from a group of n people.

Why:

Done ✓

can form a committee by:

- choosing the chair $\binom{n}{1} = n$
- from remaining $n-1$ ppl, choosing

other committee members (i.e. just choose a subset from a set of size $n-1$) 2^{n-1}

$$\Rightarrow |S| = n2^{n-1}$$

OTOH we can partition S:

$$S = A_1 \cup A_2 \cup \dots \cup A_n$$

where A_k is set of committees w/ exactly k ppl.

Before we computed:

$$|A_k| = \binom{n}{k} k$$

So

$$|S| = |A_1| + |A_2| + \dots + |A_n|$$

$$= \binom{n}{1} 1 + \binom{n}{2} 2 + \dots + \binom{n}{n} n$$

$$= \sum_{k=1}^{n} \binom{n}{k} k \checkmark$$